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The sequence reconstruction of permutations with Hamming metric

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Abstract
In the combinatorial context, one of the key problems in sequence reconstruction is to find the largest intersection of two metric balls of radius r. In this paper we study this problem for permutations of length n distorted by Hamming errors and determine the size of the largest intersection of two metric balls with radius r whose centers are at distance $d = 2, 3, 4$. Moreover, it is shown that for any $n \geqslant 3$ an arbitrary permutation is uniquely reconstructible from four distinct permutations at Hamming distance at most two from the given one, and it is proved that for any $n \geqslant 4$ an arbitrary permutation is uniquely reconstructible from $4n - 5$ distinct permutations at Hamming distance at most three from the permutation. It is also proved that for any $n \geqslant 5$ an arbitrary permutation is uniquely reconstructible from $7n^2 - 31n + 37$ distinct permutations at Hamming distance at most four from the permutation. Finally, in the case of at most r Hamming errors and sufficiently large n, it is shown that at least $\Theta(n^{r-2})$ distinct erroneous patterns are required in order to reconstruct an arbitrary permutation.

Keywords Sequence reconstruction · Permutation codes · Hamming errors

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1 Introduction

In 2001, Levenshtein [14] first proposed the sequence reconstruction problem. In this model, a sequence is transmitted through multiple channels such that all channel outputs are different and a decoder receives all the distinct outputs. Levenshtein [14, 15] studied the minimum number of transmission channels needed to exactly reconstruct the transmitted sequence. While the sequence reconstruction problem was first motivated by applications in biology and chemistry [14], this problem has received much attention because of some emerging data storage media. These storage media provide users with multiple cheap and noisy reads including DNA-based data storage [3, 13] and racetrack memories [2].

Mathematically speaking, denote by S and $\rho : S \times S \to \mathbb{N}$ a set of all sequences and a metric over the sequences in S, respectively. Assume the transmitted sequence is an arbitrary sequence in S and on every channel there are at most r errors. In this model, Levenshtein [14] proved that the minimum number of transmission channels must be greater than the largest intersection of two balls:

$$N(n, r) = \max_{x_1, x_2 \in S, x_1 \neq x_2} \{|B_r(x_1) \cap B_r(x_2)|\}, \tag{1}$$

where $B_r(x) = \{y \in S | \rho(x, y) \leqslant r\}$ is the metric ball of radius r centered at x and the length of any sequence in S is n. This means that an unknown sequence in S can be reconstructed uniquely by any set of $N(n, r) + 1$ or more distinct sequences at distance at most r from the sequence provided that such a set does exist. We call the problem of finding $N(n, r)$ as the *reconstruction problem*. In this problem, besides solving the problem stated in (1), Levenshtein [14] used the majority algorithm on each bit to successfully decode the transmitted sequence in the Hamming graph. The majority algorithm receives the estimations on each bit from every channel and simply decodes the bit according to a majority vote among all the channel estimations.

Determining the value in (1) was discussed in [14] with respect to several channels with several distances such as Hamming distance, Johnson distance, and some other distances. Later the problem was discussed in the context of permutations in [9–12, 23, 24, 26] and some other general error graphs in [16, 17]. This problem was also studied in [26] for the Grassmann graph, in [20] for insertions, and in [6, 7, 25] for deletions.

Permutation codes has received some attention in the literature due to their applications in flash memories [8], DNA storage [1, 19], and data transmission over power lines [18, 22]. In the reconstruction model, this reconstruction problem is equivalent to the required number of sequences in order to output of some sequences. In flash memories, permutation codes with the Kendall's τ-distance has been proposed in [8], and the reconstruction problem over permutations with the Kendall's τ-distance was studied in [23, 26]. In the power line application, permutation codes under Hamming distance are used to correct some transmission errors [18, 22]. In DNA storage, the rank modulation scheme (i.e., permutations) without considering distance was proposed with shotgun sequencing [1, 19]. Then, in the power line application and DNA storage, we study the reconstruction problem over permutations under Hamming distance when the transmitted sequence is not recovered. From a graph-theoretical point of view, this is the problem of reconstructing a vertex by its neighbours being at a given distance from the vertex [12]. The reconstruction problem for Cayley graphs over the permutation was studied in [9, 10] for reversal errors, in [11, 12, 17] for transposition errors, in [24] for Hamming errors.

In this paper we consider the sequence reconstruction problem of permutations distorted by Hamming errors. This paper is organised as follows. In Sect. 2, we give main definitions and notations for the sequence reconstruction problem and permutations with Hamming metric. In Sect. 3, we find the size of the maximum intersection of two metric balls with radius r whose centers are at distance $d = 2, 3, 4$. In Sect. 4, we get the exact values of $N(n, r)$ for $r = 2, 3, 4$. In particular, it is proved that that $N(n, 2) = 3$ for any $n \geqslant 3$, $N(n, 3) = 4n - 6$ for any $n \geqslant 3$, and $N(n, 4) = 7n^2 - 31n + 36$ for any $n \geqslant 4$. In Sect. 5, the lower bound on $N(n, r)$ for any $r \geqslant 5$ and $n \geqslant r$ is given and an asymptotic behaviour of $N(n, r)$ is presented. More precisely, it is shown that at least $\Theta(n^{r-2})$ distinct erroneous patterns are required in order to reconstruct an arbitrary permutation in the case of at most r Hamming errors and sufficiently large n.

2 Main definitions and notation

In this paper we follow the same notation as stated in [21, 24].

Let Sym_n, $n \geqslant 2$, be the symmetric group of permutations $\pi = [\pi_1\pi_2 \ldots \pi_n]$ written as strings in one-line notation, where $\pi_i = \pi(i)$ for any $1 \leqslant i \leqslant n$, with the identity element $I_n = [1\,2\,\ldots\,n]$. It is well-known that any permutation can be expressed as a product of disjoint cycles. For two permutations $\sigma, \pi \in \mathrm{Sym}_n$, their product $\pi \circ \sigma$ is defined as the composition of σ on π, that is, $\pi \circ \sigma(i) = \sigma(\pi(i))$ for any $i \in [n]$. For any $\pi \in \mathrm{Sym}_n$, let $disc(\pi) = [1^{h_1}2^{h_2}\ldots n^{h_n}]$ be the cycle type of π, where h_i is the number of cycles of length i. We omit components with $h_i = 0$ in $disc(\pi)$. For example, the cycle type of $\pi = (1\,2\,3)(4\,5\,6)(7\,8\,9)$ is written as $disc(\pi) = [3^3]$.

For any two permutations π and τ, Hamming distance between them is the number of positions in which these permutations differ:

$$d(\pi, \tau) = |\{i \in [n] | \pi_i \neq \tau_i\}|, \tag{2}$$

where $[n] = \{1, 2, ..., n - 1, n\}$.

Let $B_r(\pi) = \{\tau \in \mathrm{Sym}_n | d(\pi, \tau) \leqslant r\}$ and $S_r(\pi) = \{\tau \in \mathrm{Sym}_n | d(\pi, \tau) = r\}$ be a metric ball and a metric sphere of radius r centered at a permutation π. The sizes of $B_r(\pi)$ and $S_r(\pi)$ do not depend on a permutation π under Hamming distance [4]. For convenience, we put $B_r(n) = |B_r(\pi)|$ and $S_r(n) = |S_r(\pi)|$ for any $\pi \in \mathrm{Sym}_n$.

A *derangement* of order r is a permutation π with no fixed points, i.e., $\pi_i \neq i$ for any $i \in [r]$. The number D_r of distinct derangements on r elements is given by the following formula [5]:

$$D_r = r! \sum_{i=0}^{r} \frac{(-1)^i}{i!}, \tag{3}$$

where $D_0 = 1$, $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, and $D_5 = 44$.

Then the size of $S_r(n)$ is given as follows:

$$S_r(n) = \binom{n}{r} D_r,$$

and the size of $B_r(n)$ is presented as follows:

$$B_r(n) = \sum_{i=0}^{r} S_i(n) = \sum_{i=0}^{r} \binom{n}{i} D_i, \tag{4}$$

where $S_0(n) = D_0 = 1$.

For given integers d and r, let $I(n,d,r)$ be the size of the maximum intersection of two metric balls of radius r and at distance d between their centers $\pi,\tau\in \mathrm{Sym}_n$ such that:

$$I(n,d,r)=\max_{\pi,\tau\in \mathrm{Sym}_n, d(\pi,\tau)=d}|B_r(\pi)\cap B_r(\tau)|. \tag{5}$$

The formula (1) can be rewritten in terms of permutations as follows:

$$N(n,r)=\max_{\pi,\tau\in \mathrm{Sym}_n, d(\pi,\tau)\geqslant 1}|B_r(\pi)\cap B_r(\tau)|=\max_{d\geqslant 1} I(n,d,r). \tag{6}$$

Assume any permutation in Sym_n is transmitted over N channels such that there are at most r errors on each channel and all channel outputs differ from each other. Then Levenshtein [14] proved that the minimum number of channels that guarantees the existence of a decoder that successfully decode any transmitted sequence is given by $N(n,r)+1$. Based on the above definitions, we find the exact values $I(n,d,r)$ for $d=2,3,4$ and for any $r\geqslant 2$. Further, we can obtain the exact values $N(n,r)$ for $r=2,3,4$. Finally, we present the lower bound on $N(n,r)$ for any $r\geqslant 5$, and get $N(n,r)$ for any $r\geqslant 5$ and sufficiently large n.

3 Exact values $I(n,d,r)$ for $d=2,3,4$

To get main results of this section let us prove a few useful lemmas.

Lemma 1 *For any three permutations $\pi,\tau,\sigma\in \mathrm{Sym}_n$, we have:*

$$d(\pi\circ\sigma,\tau\circ\sigma)=d(\sigma\circ\pi,\sigma\circ\tau)=d(\pi,\tau)\ . \tag{7}$$

Proof Let $\pi=[\pi_1\pi_2\ldots\pi_n]$ and $\tau=[\tau_1\tau_2\ldots\tau_n]$. Any permutation $\sigma\in \mathrm{Sym}_n$ can be expressed as a product of transpositions. In the simplest case $\sigma=(i\ j)$ and we have $(i\ j)\circ\pi=[\pi_1\pi_2\ldots\pi_{i-1}\pi_j\pi_{i+1}\ldots\pi_{j-1}\pi_i\pi_{j+1}\ldots\pi_n]$. Similarly, we have $(i\ j)\circ\tau=[\tau_1\tau_2\ldots\tau_{i-1}\tau_j\tau_{i+1}\ldots\tau_{j-1}\tau_i\tau_{j+1}\ldots\tau_n]$. Clearly, it follows that $d((i\ j)\circ\pi,(i\ j)\circ\tau)=d(\pi,\tau)$. Hence, we obtain $d(\sigma\circ\pi,\sigma\circ\tau)=d(\pi,\tau)$. Moreover, we have $\pi\circ\sigma(i)=\sigma(\pi(i))$ and $\tau\circ\sigma(i)=\sigma(\tau(i))$ for any $i\in[n]$. Since σ is a permutation this means that $\pi\circ\sigma(i)$ and $\tau\circ\sigma(i)$ are distinct if and only if π_i and τ_i are distinct. Thus, we have that $d(\pi\circ\sigma,\tau\circ\sigma)=d(\pi,\tau)$. The case when σ is presented by a product of a few transpositions is proved in a similar way. □

Below we use the result obtained in [24].

Lemma 2 [24, Lemma 7] *For any two permutations $\pi,\tau\in \mathrm{Sym}_n$, if $disc(\pi)=disc(\tau)$ then we have:*

$$d(I_n,\pi)=d(I_n,\tau)\ \textit{and}\ |B_r(\pi)\cap B_r(I_n)|=|B_r(\tau)\cap B_r(I_n)|$$

for any $n\geqslant r$.

Let $\pi\circ B=\{\pi\circ\beta|\beta\in B\}$ and $B\circ\pi=\{\beta\circ\pi|\beta\in B\}$ for any permutation $\pi\in \mathrm{Sym}_n$ and for any subset $B\subset \mathrm{Sym}_n$.

Lemma 3 *For any two permutations $\pi,\tau\in \mathrm{Sym}_n$ with $d(\pi,\tau)=d$ there exists a permutation σ such that $|B_r(\pi)\cap B_r(\tau)|=|B_r(\sigma)\cap B_r(I_n)|$, where $d(\sigma,I_n)=d$ and $\sigma_i=i$ for any $r\geqslant 2$ and any $i\geqslant d+1$. If $\lfloor\frac{d-1}{2}\rfloor+1\leqslant r\leqslant n$ then there exists a permutation σ such that $I(n,d,r)=|B_r(\sigma)\cap B_r(I_n)|$, where $d(\sigma,I_n)=d$ and $\sigma_i=i$ with $d+1\leqslant i\leqslant n$.*

Proof By the definition, left multiplication by a transposition $(i\ j)$ exchanges the elements of a permutation π in positions i and j, and right multiplication by $(i\ j)$ exchanges the elements i, j of π. Hence, using the property of left multiplication, there exists a permutation α such that $\alpha \circ \pi(i) \neq \alpha \circ \tau(i)$ for any $i \in [d]$, and $\alpha \circ \pi(i) = \alpha \circ \tau(i)$ with $d+1 \leqslant i \leqslant n$. Using the property of right multiplication, we have $I_n(i) \neq \alpha \circ \tau \circ (\alpha \circ \pi)^{-1}(i)$ for any $i \in [d]$, and $I_n(i) = \alpha \circ \tau \circ (\alpha \circ \pi)^{-1}(i)$ with $d+1 \leqslant i \leqslant n$. Let $\sigma = \alpha \circ \tau \circ (\alpha \circ \pi)^{-1}$ then by Lemma 1 we have $\alpha \circ \big(B_r(\pi) \cap B_r(\tau)\big) \circ (\alpha \circ \pi)^{-1} = B_r(\sigma) \cap B_r(I_n)$. Thus, $|B_r(\pi) \cap B_r(\tau)| = |B_r(\sigma) \cap B_r(I_n)|$, where $d(\sigma, I_n) = d$ and $\sigma_i = i$ for any $i \geqslant d+1$.

Moreover, under assumption that $I(n,d,r) = |B_r(\pi) \cap B_r(\tau)|$ and $\lfloor \frac{d-1}{2} \rfloor + 1 \leqslant r \leqslant n$, there exists a permutation σ such that $I(n,d,r) = |B_r(\sigma) \cap B_r(I_n)|$ where $d(\sigma, I_n) = d$ and $\sigma_i = i$ for $d+1 \leqslant i \leqslant n$. □

Now we are ready to get the exact value $I(n,2,r)$ for any $r \geqslant 2$.

Theorem 1 *For any integer r, $2 \leqslant r \leqslant n$, the following holds:*

$$I(n,2,r) = \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2}\big). \tag{8}$$

Proof Let $\sigma \in \mathrm{Sym}_n$ and $d(\sigma, I_n) = 2$. Since $d = 2$ then $disc(\sigma) = [2^1]$. For $\sigma = (1\,2) = [2\,1\,3\,\ldots\,n]$ by Lemmas 2 and 3 we have:

$$I(n,2,r) = |B_r(I_n) \cap B_r(\sigma)|.$$

Let $\alpha \in B_r(I_n)$ and $d(\alpha, I_n) = t$, where $0 \leqslant t \leqslant r$, and let us check whether $\alpha \in B_r(\sigma)$. For convenience, we put $\alpha = [\alpha_1\,\alpha_2\,\alpha_3\,\ldots\,\alpha_n] = [a_1\,a_2]$, where $a_1 = [\alpha_1\,\alpha_2]$ and $a_2 = [\alpha_3\,\ldots\,\alpha_n]$. In a similar way, we put $I_n = [e_1\,e_2]$ and $\sigma = [\sigma_1\,\sigma_2]$, where $e_1 = [1\,2]$, $\sigma_1 = [2\,1]$ and $e_2 = [3\,\ldots\,n] = \sigma_2$. Thus, $0 \leqslant d(a_1, e_1) \leqslant 2$. For $t = 0, 1, 2$, let us find the size of the set:

$$\{\alpha \in \mathrm{Sym}_n | \alpha \in B_r(I_n) \cap B_r(\sigma), d(a_1, e_1) = t\}.$$

If $d(a_1, e_1) = 0$ then $\alpha_1 = 1, \alpha_2 = 2$ which gives $d(a_1, \sigma_1) = 2$. Since $d(\alpha, I_n) = t$ this means that $d(a_2, e_2) = d(a_2, \sigma_2) = t$. Thus, $d(\alpha, \sigma) = t + 2$, and $\alpha \in B_r(\sigma)$ for any t, where $0 \leqslant t \leqslant r-2$. Hence, by (4), the number of such permutations is $\sum_{t=0}^{r-2} \binom{n-2}{t} D_t$.

If $d(a_1, e_1) = 1$ then either $\alpha_1 = 1, \alpha_2 \neq 2$, or $\alpha_1 \neq 1, \alpha_2 = 2$. If $\alpha_1 = 1$ and $\alpha_2 \neq 2$ then $d(a_1, \sigma_1) = 2$. Since $d(\alpha, I_n) = t$ this means that $d(a_2, e_2) = d(a_2, \sigma_2) = t - 1$. Hence, $d(\alpha, \sigma) = t + 1$. Similarly, if $\alpha_1 \neq 1$ and $\alpha_2 = 2$ we also have $d(\alpha, \sigma) = t + 1$. Thus, $\alpha \in B_r(\sigma)$ for any t such that $1 \leqslant t \leqslant r - 1$, and by (4), the number of such permutations is $2\sum_{t=1}^{r-1} \binom{n-2}{t-1} D_t$.

If $d(a_1, e_1) = 2$ then $d(a_1, \sigma_1) \leqslant 2$. Since $d(\alpha, I_n) = t$ then $d(\alpha, \sigma) \leqslant t$. Hence, $\alpha \in B_r(\sigma)$ for any t, where $2 \leqslant t \leqslant r$, and by (4), the number of these permutations is $\sum_{t=2}^{r} \binom{n-2}{t-2} D_t$.

Thus, taking into account the cases above we get the sought result:

$$\begin{aligned} I(n,2,r) =& |B_r(\sigma) \cap B_r(I_n)| \\ =& \sum_{t=0}^{r-2} \binom{n-2}{t} D_t + 2\sum_{t=1}^{r-1} \binom{n-2}{t-1} D_t + \sum_{t=2}^{r} \binom{n-2}{t-2} D_t \\ =& \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2}\big). \end{aligned}$$

□

In particular, by Theorem 1 and (3), the following equations hold:

$$I(n,2,2) = 2 \text{ for any } n \geqslant 2, \tag{9}$$
$$I(n,2,3) = 4n-6 \text{ for any } n \geqslant 3, \tag{10}$$
$$I(n,2,4) = 7n^2-31n+36 \text{ for any } n \geqslant 4. \tag{11}$$

Now let us obtain the exact value $I(n,3,r)$ for any $r \geqslant 2$. In what follows below, we define the following set:

$$S_r^j(I_r) = \{\alpha \in \mathrm{Sym}_r | \alpha(j) = 2, \alpha(i) \neq i \text{ for any } i \in [r]\}, \tag{12}$$

where $j \in [r]\backslash\{2\}$.

Lemma 4 *For any $r \geqslant 2$ and any $j \in [r]\backslash\{2\}$, we have:*

$$|S_r^j(I_r)| = \frac{1}{r-1} D_r. \tag{13}$$

Proof By (12), it follows that $S_r(I_r) = \cup_{j\in[r]\backslash\{2\}} S_r^j(I_r)$. When $r=2$ then $|S_r^1(I_r)| = D_r$. When $r \geqslant 3$ we prove that $|S_r^1(I_r)| = |S_r^j(I_r)|$ for any $3 \leqslant j \leqslant r$. For convenience, let $A_1 = \{\alpha \in S_r^1(I_r) | \alpha(j) \neq 1\}$, $A_2 = \{\alpha \in S_r^1(I_r) | \alpha(j) = 1\}$, $B_1 = \{\alpha \in S_r^j(I_r) | \alpha(1) \neq j\}$, $B_2 = \{\alpha \in S_r^j(I_r) | \alpha(1) = j\}$ for some $3 \leqslant j \leqslant r$. By the definition of $S_r^1(I_r)$ and $S_r^j(I_r)$, it is easily verified that $S_r^1(I_r) = A_1 \cup A_2$ and $S_r^j(I_r) = B_1 \cup B_2$. Then $(1\ j) \circ A_1 = B_1$ and $A_2 \circ (j\ 2) \circ (2\ 1) = B_2$. Thus, $|A_1| = |B_1|$ and $|A_2| = |B_2|$. Then, we have $|S_r^1(I_r)| = |S_r^j(I_r)|$ for any $3 \leqslant j \leqslant r$. Since $S_r^i(I_r) \cap S_r^j(I_r) = \emptyset$ for any two distinct $i, j \in [r]\backslash\{2\}$, this means that $|S_r(I_r)| = \sum_{j\in[r]\backslash\{2\}} |S_r^j(I_r)| = (r-1)|S_r^j(I_r)| = D_r$ for any $j \in [r]\backslash\{2\}$ which immediately gives us (13). □

Theorem 2 *For any integer r, $3 \leqslant r \leqslant n$, the following holds:*

$$I(n,3,r) = \sum_{t=0}^{r-3} \left(\binom{n-3}{t} \left(D_t + 3(D_{t+1} + D_{t+2}) + D_{t+3} \right) \right) + \frac{3}{r-1} \binom{n-3}{r-2} D_r. \tag{14}$$

Moreover, for any $n \geqslant 3$, we have $I(n,3,2) = 3$.

Proof Let $\sigma \in \mathrm{Sym}_n$ and $d(\sigma, I_n) = 3$. Since $d = 3$ then $disc(\sigma) = [3^1]$. For $\sigma = (1\,2\,3) = [2\,3\,1\,4\,\ldots\,n]$ by Lemmas 2 and 3, we have:

$$I(n,3,r) = |B_r(\sigma) \cap B_r(I_n)|.$$

What it follows below, we use the same technique as it was used to prove Theorem 1. We consider $\alpha \in B_r(I_n)$ with $d(\alpha, I_n) = t$, where $0 \leqslant t \leqslant r$, and check whether $\alpha \in B_r(\sigma)$. We put $\alpha = [\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\,\ldots\,\alpha_n] = [a_1\,a_2]$ with $a_1 = [\alpha_1\,\alpha_2\,\alpha_3]$ and $a_2 = [\alpha_4\,\ldots\,\alpha_n]$. In a similar way, we consider $I_n = [e_1\,e_2]$ and $\sigma = [\sigma_1\,\sigma_2]$, where $e_1 = [1\,2\,3]$, $\sigma_1 = [2\,3\,1]$ and $e_2 = [4\,\ldots\,n] = \sigma_2$.

To find $I(n,3,r)$, the following cases are considered.

Case 1 ($r=2$): If $d(a_1, e_1) = 0$ then $d(\alpha, \sigma) \geqslant d(a_1, \sigma_1) = 3$ which means $\alpha \notin B_2(\sigma)$. If $d(a_1, e_1) = 1$ then $d(a_2, \sigma_2) = d(a_2, e_2) \geqslant 1$ and $d(a_1, \sigma_1) \geqslant 2$. Thus, $d(\alpha, \sigma) = d(a_1, \sigma_1) + d(a_2, \sigma_2) \geqslant 3$, and $\alpha \notin B_2(\sigma)$. If $d(a_1, e_1) = 2$ and $d(a_2, e_2) = 0$ then $\alpha \in$

$\{[1\,3\,2\,e_2], [3\,2\,1\,e_2], [2\,1\,3\,e_2]\}$, and for any such α we have $d(\alpha,\sigma)=2$, hence $I(n,3,2)=|B_2(\sigma)\cap B_2(I_n)|=3$ in this case.

Case 2 ($r\geqslant 3$): If $d(a_1,e_1)=0$ then $d(a_1,\sigma_1)=3$, and since $d(\alpha,I_n)=t$ it follows that $d(a_2,e_2)=d(a_2,\sigma_2)=t$. Thus, $d(\alpha,\sigma)=t+3$ which means $\alpha\in B_r(\sigma)$ for any t, where $0\leqslant t\leqslant r-3$. By (4), the number of these permutations is $\sum_{t=0}^{r-3}\binom{n-3}{t}D_t$. Similarly, if $d(a_1,e_1)=1$ and $d(a_1,e_1)=3$ then the corresponding numbers of permutations $\alpha\in B_r(\sigma)$ are given by $3\sum_{t=1}^{r-2}\binom{n-3}{t-1}D_t$ and $\sum_{t=3}^{r}\binom{n-3}{t-3}D_t$, respectively.

If $d(a_1,e_1)=2$ then either $d(a_1,\sigma_1)=3$ or $d(a_1,\sigma_1)=2$, and since $d(\alpha,I_n)=t$ then $d(a_2,e_2)=d(a_2,\sigma_2)=t-2$. If $2\leqslant t\leqslant r-1$ then we have $d(\alpha,\sigma)\leqslant r$, and the number of these permutations is $3\sum_{t=2}^{r-1}\binom{n-3}{t-2}D_t$. If $t=r$ and $d(a_1,\sigma_1)=2$ then $d(\alpha,\sigma)=r$. Hence, $\alpha\in B_r(\sigma)$. Choosing two elements of $\{1,2,3\}$ and $r-2$ elements of $\{4,\ldots,n\}$ we get the number of sought permutations as $3\binom{n-3}{r-2}$. Without loss of generality, let $a_1=[1\,3\,s]$, where $s\in\{2,4,5,\ldots,r+1\}$, with $\alpha_i\neq i$, $2\leqslant i\leqslant r+1$, and $\alpha_i=i$ for $r+2\leqslant i\leqslant n$. By Lemma 4, the number of permutations α satisfying the conditions above is $\frac{1}{r-1}D_r$. If $t=r$ and $d(a_1,\sigma_1)=2$ the number of permutations α is $\frac{3}{r-1}\binom{n-3}{r-2}D_r$. If $t=r$ and $d(a_1,\sigma_1)=3$ then $d(\alpha,\sigma)=r+1$ and $\alpha\notin B_r(\sigma)$. Thus, totally in this case the number of sought permutations is $3\sum_{t=2}^{r-1}\binom{n-3}{t-2}D_t+\frac{3}{r-1}\binom{n-3}{r-2}D_r$.

Finally, for any $r\geqslant 3$ we have:

$$\begin{aligned}I(n,3,r)=&|B_r(\sigma)\cap B_r(I_n)|=\sum_{t=0}^{r-3}\binom{n-3}{t}D_t+3\sum_{t=1}^{r-2}\binom{n-3}{t-1}D_t+\\&+3\sum_{t=2}^{r-1}\binom{n-3}{t-2}D_t+\frac{3}{r-1}\binom{n-3}{r-2}D_r+\sum_{t=3}^{r}\binom{n-3}{t-3}D_t\\=&\sum_{t=0}^{r-3}\left(\binom{n-3}{t}\left(D_t+3(D_{t+1}+D_{t+2})+D_{t+3}\right)\right)+\frac{3}{r-1}\binom{n-3}{r-2}D_r.\end{aligned}$$

□

In particular, by Theorem 2 and (3), we have:

$$I(n,3,2)=3 \quad \text{for any } n\geqslant 3, \tag{15}$$

$$I(n,3,3)=3(n-1) \quad \text{for any } n\geqslant 3, \tag{16}$$

$$I(n,3,4)=\frac{9n^2-27n+12}{2} \quad \text{for any } n\geqslant 4. \tag{17}$$

Now our goal is to find the exact value of $I(n,4,r)$ for any $r\geqslant 2$. Below we consider the following two permutations:

$$\alpha^{(1)}=(1\,2\,3\,4)=[2\,3\,4\,1\,5\,\ldots\,n] \tag{18}$$

and

$$\alpha^{(2)}=(1\,2)(3\,4)=[2\,1\,4\,3\,5\,\ldots\,n] \tag{19}$$

where $\alpha^{(j)}(i)=i$ for any i such that $5\leqslant i\leqslant n$ and for $j=1,2$.

Lemma 5 *For any $n\geqslant 4$ and $r\geqslant 2$, we have:*

$$I(n,4,r)=\max_{j\in[2]}\{|B_r(\alpha^{(j)})\cap B_r(I_n)|\}, \tag{20}$$

where $\alpha^{(j)}$ are defined by (18) and (19).

Proof Let $\sigma \in \mathrm{Sym}_n$ and $d(\sigma, I_n) = 4$. By Lemma 3, there exists a permutation α such that $I(n, 4, r) = |B_r(\sigma) \cap B_r(I_n)|$, where $\sigma(i) = i$ for any $i \geqslant 5$, and $d(\alpha, I_n) = 4$. Since $d = 4$ then either $disc(\sigma) = [4^1]$ or $[2^2]$. By Lemma 2, we immediately obtain (20). □

Let us put $D_{-1} = 0$ while getting results on D_r.

Lemma 6 *For any $r \geqslant 3$, we have:*

$$D_r = r \cdot D_{r-1} + (-1)^r,$$

and $\frac{2}{r-1} D_r - D_{r-2} - D_{r-3} > 0$.

Proof By (3), we immediately get $D_r = r D_{r-1} + (-1)^r$. If $r \geqslant 3$ it is easily shown that $D_r \geqslant 2$. Hence, we have $D_r - D_{r-1} \geqslant (r-1) D_{r-1} - 1 > 0$ for any $r \geqslant 3$. Therefore, we have:

$$\begin{aligned} \frac{2}{r-1} D_r - D_{r-2} - D_{r-3} &\geqslant \frac{2}{r-1} (r D_{r-1} - 1) - D_{r-2} - D_{r-3} \\ &= (D_{r-1} - D_{r-2}) + (D_{r-1} - D_{r-3}) + \frac{2}{r-1} (D_{r-1} - 1) \\ &> 0, \end{aligned}$$

which completes the proof. □

One more technical result is required to get $I(n, 4, r)$. We consider the following set of permutations:

$$\mathcal{A} = \{\sigma \in \mathrm{Sym}_n |\, \sigma_1 = 2, \sigma_2 = 3, \sigma_3 \neq 1, \sigma_4 = 4, d(\sigma, I_n) = r\}.$$

Lemma 7 *For any integer r such that $4 \leqslant r \leqslant n-1$, we have:*

$$|\mathcal{A}| = \binom{n-4}{r-3} D_{r-2}. \tag{21}$$

Proof Let $\sigma = [\sigma_1\, \sigma_2\, \sigma_3\, \sigma_4\, \sigma_5\, \ldots\, \sigma_n] = [s_1\, s_2]$, where $s_1 = [\sigma_1\, \sigma_2\, \sigma_3\, \sigma_4]$, $s_2 = [\sigma_5\, \ldots\, \sigma_4]$, and let $I_n = [e_1\, e_2]$, where $e_1 = [1\,2\,3\,4]$, $e_2 = [5\, \ldots\, n]$. Since $\sigma \in \mathcal{A}$ this means that $d(\sigma_2, e_2) = r - 3$ and $\sigma_3 = i$ for some $i \in [n] \backslash [4]$. Thus, considering the set $\{1, 2, 3, 4\}$ as an entity we get a derangement of order $r - 2$ and the equality (21). □

Now let us give the exact values $I(n, 4, r)$ for some n and r.

Lemma 8 *For any $n \geqslant 4$, $I(n, 4, 2) = 2$, $I(n, 4, 3) = 10$, and $I(n, 4, n) = n!$.*

Proof By Lemma 5, it follows that $I(n, 4, r) = \max_{j \in [2]} \{|B_r(I_n) \cap B_r(\alpha^{(j)})|\}$, where the permutations $\alpha^{(j)}$ are given by (18) and (19). To get the sought results, we use the technique from Theorem 1. Let $\sigma = [s_1\, s_2] \in B_r(I_n)$, where $s_1 = [\sigma_1\, \sigma_2\, \sigma_3\, \sigma_4]$ and $s_2 = [\sigma_5\, \ldots\, \sigma_n]$. In a similar way, we put $I_n = [e_1\, e_2]$ with $e_1 = [1\,2\,3\,4]$ and $e_2 = [5\, \ldots\, n]$. We also put $\alpha^{(1)} = [a_1^{(1)}\, a_2]$ and $\alpha^{(2)} = [a_1^{(2)}\, a_2]$, where $a_1^{(1)} = [2\,3\,4\,1]$, $a_1^{(2)} = [2\,1\,4\,3]$, and $a_2 = [5\, \ldots\, n] = e_2$.

Case 1 ($r = 2$): Let us check whether $\sigma \in B_2(\alpha^{(1)})$. If either $d(e_1, s_1) = 0$ or $d(e_1, s_1) = 1$ then $d(\sigma, \alpha^{(1)}) \geqslant d(s_1, a_1^{(1)}) \geqslant 3$. Hence, $\sigma \notin B_2(\alpha^{(1)})$. If $d(e_1, s_1) = 2$ then $d(s_1, a_1^{(1)}) \geqslant 3$ because no transpositions are appeared neither in I_n nor in $\alpha^{(1)}$. Hence, $|B_2(I_n) \cap B_2(\alpha^{(1)})| = 0$. Similarly, we discuss whether $\sigma \in B_2(\alpha^{(2)})$. If either $d(e_1, s_1) = 0$ or $d(e_1, s_1) = 1$ then $d(\sigma, \alpha^{(2)}) \geqslant d(s_1, a_1^{(2)}) \geqslant 3$. If $d(e_1, s_1) = 2$ then $s_1 = [2\,1\,3\,4]$ or $s_1 = [1\,2\,4\,3]$ if and only if $d(s_1, a_1^{(2)}) = 2$. Since $\sigma \in B_2(I_n)$ then $s_2 = e_2$. Thus, $[2\,1\,3\,4\,e_2], [1\,2\,4\,3\,e_2] \in B_2(\alpha^{(2)})$. So, we prove that $|B_2(I_n) \cap B_2(\alpha^{(2)})| = 2$. Therefore, $I(n, 4, 2) = 2$.

Case 2 ($r = 3$): There exist two subcases.

First, we check whether $\sigma \in B_3(\alpha^{(1)})$. If $\sigma \in B_3(I_n) \cap B_3(\alpha^{(1)})$ then $d(s_2, e_2) \in \{0, 1, 2, 3\}$. If $d(s_2, e_2) \geqslant 2$ then $d(s_2, a_2) = d(s_2, e_2) \geqslant 2$ and $d(s_1, e_1) \leqslant 1$. Since $d(s_1, e_1) \leqslant 1$ and $d(a_1^{(1)}, e_1) = 4$ this means that $d(s_1, a_1^{(1)}) \geqslant 3$ and, hence we have $d(\sigma, \alpha^{(1)}) \geqslant 5$.

If $d(s_2, e_2) = 1$ then $d(s_2, a_2) = d(s_2, e_2) = 1$ and $d(s_1, e_1) \leqslant 2$. If $d(s_1, e_1) \leqslant 1$ then it is easy to see that $d(s_1, a_1^{(1)}) \geqslant 3$ and $d(\sigma, \alpha^{(1)}) \geqslant 4$. If $d(s_1, e_1) = 2$ then since $d(s_2, e_2) = 1$ it follows that $s_1(i) = j \neq e_1(i)$ for some $i \in [4]$ and $j \geqslant 5$, and we have $s_1(i) = j \neq a_1^{(1)}(i)$ and $d(s_1, a_1^{(1)}) \geqslant 3$ which again gives us $d(\sigma, \alpha^{(1)}) \geqslant 4$.

Thus, for any $\sigma \in B_3(I_n) \cap B_3(\alpha^{(1)})$ we have $d(s_2, e_2) = d(s_2, a_2) = 0$. Now we consider the set:

$$B_3^{'}(I_n) = \{\sigma \,|\, \sigma \in B_3(I_n) \text{ and } \sigma(i) = i \text{ for any } 5 \leqslant i \leqslant n\},$$

for which by (3), we have $|B_3^{'}(I_n)| = 24 - D_4 = 15$. Therefore, for any $\beta \in B_3^{'}(I_n)$, we have $d(\beta, \alpha^{(1)}) \leqslant 4$ and we consider one more set:

$$S_4^{'}(\alpha^{(1)}) = \{\beta \,|\, d(\beta, \alpha^{(1)}) = 4 \ \text{ and } \beta(i) = i \text{ for any } 5 \leqslant i \leqslant n\},$$

for which by (3), $|S_4^{'}(\alpha^{(1)})| = 9$. It is easy to check that the set $S_4^{'}(\alpha^{(1)})$ contains the following permutations: $I_n, [1\,4\,3\,2\,e_2], [1\,4\,2\,3\,e_2], [3\,4\,1\,2\,e_2], [3\,2\,1\,4\,e_2], [3\,1\,2\,4\,e_2], [4\,1\,3\,2\,e_2], [4\,1\,2\,3\,e_2], [4\,2\,1\,3\,e_2]$. Thus, $|S_4^{'}(\alpha^{(1)}) \cap B_3^{'}(I_n)| = 7$, and finally, for the permutation $\alpha^{(1)}$ we have $|B_3(I_n) \cap B_3(\alpha^{(1)})| = |B_3^{'}(I_n) \setminus S_4^{'}(\alpha^{(1)})| = |B_3^{'}(I_n)| - |S_4^{'}(\alpha^{(1)}) \cap B_3^{'}(I_n)| = 15 - 7 = 8$.

Second, we check whether $\sigma \in B_3(\alpha^{(2)})$. If $\sigma \in B_3(I_n) \cap B_3(\alpha^{(2)})$ then $d(s_2, e_2) \in \{0, 1, 2, 3\}$. If $d(s_2, e_2) \geqslant 2$ then $d(s_2, a_2) \geqslant 2$ and $d(s_1, e_1) \leqslant 1$. Hence we have $d(\sigma, \alpha^{(2)}) \geqslant 5$.

If $d(s_2, e_2) = 1$ then $d(s_2, a_2) = 1$ and $d(s_1, e_1) \leqslant 2$. If $d(s_1, e_1) \leqslant 1$ then $d(s_1, a_1^{(2)}) \geqslant 3$ and $d(\sigma, \alpha^{(2)}) \geqslant 4$. If $d(s_1, e_1) = 2$ then since $d(s_2, e_2) = 1$ it follows that $s_1(i) = j \neq e_1(i)$ for some $i \in [4]$ and $j \geqslant 5$, and we have $s_1(i) = j \neq a_1^{(2)}(i)$ and $d(s_1, a_1^{(2)}) \geqslant 3$ which again gives us $d(\sigma, \alpha^{(2)}) \geqslant 4$. Thus, for any $\sigma \in B_3(I_n) \cap B_3(\alpha^{(2)})$ we have $d(s_2, e_2) = d(s_2, a_2) = 0$. We consider the following set:

$$S_4^{'}(\alpha^{(2)}) = \{\beta \,|\, d(\beta, \alpha^{(2)}) = 4 \text{ and } \beta(i) = i \text{ for any } 5 \leqslant i \leqslant n\},$$

of cardinality $|S_4^{'}(\alpha^{(2)})| = 9$ which contains the following permutations: $I_n, [1\,4\,3\,2\,e_2], [1\,3\,2\,4\,e_2], [3\,2\,1\,4\,e_2], [3\,4\,1\,2\,e_2], [3\,4\,2\,1\,e_2], [4\,3\,1\,2\,e_2], [4\,3\,2\,1\,e_2], [4\,2\,3\,1\,e_2]$. Hence, $|S_4^{'}(\alpha^{(2)}) \cap B_3^{'}(I_n)| = 5$ and $|B_3(I_n) \cap B_3(\alpha^{(2)})| = |B_3^{'}(I_n) \setminus S_4^{'}(\alpha^{(2)})| = |B_3^{'}(I_n)| - |S_4^{'}(\alpha^{(2)}) \cap B_3^{'}(I_n)| = 15 - 5 = 10$.

Thus, $I(n, 4, 3) = \max_{j \in [2]}\{|B_3(I_n) \cap B_3(\alpha^{(j)})|\} = 10$.

The case $r = n$ is trivial so we immediately get the result as $I(n, 4, n) = n!$ which completes the proof. □

Now we are ready to get the exact value $I(n, 4, r)$ for any $r \geqslant 4$.

Theorem 3 *For any $r \geqslant 4$, the following holds:*

$$I(n, 4, r) = \sum_{t=0}^{r-4} \left(\binom{n-4}{t} \Big(D_t + 4D_{t+1} + 6D_{t+2} + 4D_{t+3} + D_{t+4} \Big) \right) + \Delta, \qquad (22)$$

where

$$\Delta = 2 \cdot \max\left\{\left(\binom{n-4}{r-2}D_{r-2} - \binom{n-4}{r-3}D_{r-3}\right), 0\right\}$$
$$+\binom{n-4}{r-3}\frac{4}{r-2}D_{r-1} + 4\binom{n-4}{r-3}\left(\frac{2}{r-1}D_r - D_{r-2} - D_{r-3}\right).$$

Proof By Lemma 5, $I(n,4,r) = \max_{j\in[2]}\{|B_r(I_n)\cap B_r(\alpha^{(j)})|\}$, where $\alpha^{(j)}$ are given by (18) and (19). Let $\sigma = [s_1\, s_2] \in B_r(I_n)$, where $s_1 = [\sigma_1\,\sigma_2\,\sigma_3\,\sigma_4]$, $s_2 = [\sigma_5 \ldots \sigma_n]$, and $I_n = [e_1\, e_2]$ with $e_1 = [1\,2\,3\,4]$, $e_2 = [5 \ldots n]$. Let $\alpha^{(1)} = [a_1^{(1)}\, a_2]$, $\alpha^{(2)} = [a_1^{(2)}\, a_2]$, where $a_1^{(1)} = [2\,3\,4\,1]$, $a_1^{(2)} = [2\,1\,4\,3]$, $a_2 = e_2$.

Case 1: Let us compute $|B_r(I_n)\cap B_r(\alpha^{(1)})|$. Let $\sigma \in B_r(I_n)\cap B_r(\alpha^{(1)})$ and $d(\sigma, I_n) = t$. If $d(s_1, e_1) = 0$ then $d(s_1, a_1^{(1)}) = 4$ and $d(\sigma, \alpha^{(1)}) = t + 4$. Since $d(\sigma, \alpha^{(1)}) \leqslant r$ it follows that $t \leqslant r - 4$. Hence, we have:

$$|\{\sigma \in B_r(I_n)\cap B_r(\alpha^{(1)})|\, d(s_1, e_1) = 0\}| = \sum_{t=0}^{r-4}\binom{n-4}{t}D_t. \tag{23}$$

If $d(s_1, e_1) = 1$ then $d(s_1, a_1^{(1)}) = 4$, $d(s_2, a_2) = d(s_2, e_2) = t - 1$. Thus, $d(\sigma, \alpha^{(1)}) = t+3$ and we have:

$$|\{\sigma \in B_r(I_n)\cap B_r(\alpha^{(1)})|\, d(s_1, e_1) = 1\}| = 4\sum_{t=1}^{r-3}\binom{n-4}{t-1}D_t. \tag{24}$$

Similarly, if $d(s_1, e_1) = 4$ we have:

$$|\{\sigma \in B_r(I_n)\cap B_r(\alpha^{(1)})|\, d(s_1, e_1) = 4\}| = \sum_{t=4}^{r}\binom{n-4}{t-4}D_t. \tag{25}$$

If $d(s_1, e_1) = 2$ then either $d(s_1, a_1^{(1)}) = 4$ or $d(s_1, a_1^{(1)}) = 3$. Choosing any two distinct elements from the set $\{1, 2, 3, 4\}$ gives six choices to get s_1 such that $d(s_1, e_1) = 2$. Without loss of generality, let us take 1 and 2 such that $\sigma_1 \neq 1$ and $\sigma_2 \neq 2$. Hence, if $\sigma_1 \neq 2$ then $d(s_1, a_1^{(1)}) = 4$ and we have $d(\sigma, \alpha^{(1)}) = t + 2$. If $\sigma_1 = 2$ then $d(s_1, a_1^{(1)}) = 3$ and $d(\sigma, \alpha^{(1)}) = t + 1$. If $t \leqslant r - 2$ we always have $d(\sigma, \alpha^{(1)}) \leqslant r$. Thus, we have:

$$|\{\sigma \in B_r(I_n)\cap B_r(\alpha^{(1)})|\, d(s_1, e_1) = 2,\ \sigma(3) = 3,\ \sigma(4) = 4,\ d(\sigma, I_n) \leqslant r-2\}|$$
$$= \sum_{t=2}^{r-2}\binom{n-4}{t-2}D_t.$$

If $\sigma_1 = 2$ and $t = r - 1$ then $d(\sigma, \alpha^{(1)}) = r$ and, by Lemma 4, we have:

$$|\{\sigma \in B_r(I_n)\cap B_r(\alpha^{(1)})|\, d(s_1, e_1) = 2,\ \sigma_1 = 2,\ \sigma_3 = 3,\ \sigma_4 = 4,\ d(\sigma, I_n) = r-1\}|$$
$$= \frac{1}{r-2}\binom{n-4}{r-3}D_{r-1}.$$

Therefore, via choosing 1, 2 the number of permutations σ satisfying the above conditions is $\sum_{t=2}^{r-2}\binom{n-4}{t-2}D_t + \frac{1}{r-2}\binom{n-4}{r-3}D_{r-1}$. We get the same number if choosing 1, 4 or 2, 3 or 3, 4 since in all these cases we have either $d(s_1, a_1^{(1)}) = 4$ or $d(s_1, a_1^{(1)}) = 3$. However, choosing 1, 3 or 2, 4 we have $d(s_1, a_1^{(1)}) = 4$. Hence, the number of sought permutations in these cases is $\sum\limits_{t=2}^{r-2} \binom{n-4}{t-2} D_t$.

Thus, if $d(s_1, e_1) = 2$ then totally we have:

$$|\{\sigma \in B_r(I_n) \cap B_r(\alpha^{(1)}) | d(s_1, e_1) = 2\}| = 6 \sum_{t=2}^{r-2} \binom{n-4}{t-2} D_t + \frac{4}{r-2} \binom{n-4}{r-3} D_{r-1}. \tag{26}$$

If $d(s_1, e_1) = 3$ then $d(s_1, a_1^{(1)}) \in \{2, 3, 4\}$. Choosing any three elements from the set $\{1, 2, 3, 4\}$ gives four choices to get s_1 with $d(s_1, e_1) = 3$. Without loss of generality, we take $1, 2, 3$ such that $\sigma_i \neq i$, $i \in [3]$, and compute the size of the set $\{\sigma \,|\, d(s_1, e_1) = 3,\ \sigma \in B_r(I_n) \cap B_r(\alpha^{(1)})\}$. If $\sigma_1 \neq 2$ and $\sigma_2 \neq 3$ then $d(s_1, a_1^{(1)}) = 4$ and $d(\sigma, \alpha^{(1)}) = t + 1$. If $\sigma_1 = 2$ or $\sigma_2 = 3$ then either $d(s_1, a_1^{(1)}) = 3$ or $d(s_1, a_1^{(1)}) = 2$, and $d(\sigma, \alpha^{(1)}) \leqslant t$.

Thus, if $d(s_1, e_1) = 3$ and $d(\sigma, I_n) \leqslant r - 1$ then we have:

$$|\{\sigma \in B_r(I_n) \cap B_r(\alpha^{(1)}) |\, \sigma_4 = 4, d(s_1, e_1) = 3, d(\sigma, I_n) \leqslant r - 1\}| = \sum_{t=3}^{r-1} \binom{n-4}{t-3} D_t. \tag{27}$$

Moreover, by Lemmas 4 and 7, if $d(s_1, e_1) = 3$ and $d(\sigma, I_n) = r$ then we have:

$$|\{\sigma \,|\, \sigma_1 = 2,\ \sigma_4 = 4\}| = \binom{n-4}{r-3} \frac{D_r}{r-1},$$

$$|\{\sigma \,|\, \sigma_2 = 3,\ \sigma_4 = 4\}| = \binom{n-4}{r-3} \frac{D_r}{r-1},$$

$$|\{\sigma \,|\, \sigma_3 = 1,\ \sigma_1 = 2,\ \sigma_2 = 3,\ \sigma_4 = 4\}| = \binom{n-4}{r-3} D_{r-3},$$

and

$$|\{\sigma \,|\, \sigma_3 \neq 1,\ \sigma_1 = 2,\ \sigma_2 = 3,\ \sigma_4 = 4\}| = \binom{n-4}{r-3} D_{r-2},$$

which all together gives us:

$$\begin{aligned} &|\{\sigma \in B_r(I_n) \cap B_r(\alpha^{(1)}) |\, \sigma_1 = 2 \text{ or } \sigma_2 = 3, \sigma_4 = 4, d(s_1, e_1) = 3, d(\sigma, I_n) = r\}| \\ &\qquad = \binom{n-4}{r-3} \Big(\frac{2}{r-1} D_r - D_{r-2} - D_{r-3} \Big). \end{aligned} \tag{28}$$

Thus, if $d(s_1, e_1) = 3$ then by (27) and (28) totally we have:

$$\begin{aligned} |\{\sigma \in B_r(I_n) \cap B_r(\alpha^{(1)}) |\, d(s_1, e_1) = 3\}| &= 4 \sum_{t=3}^{r-1} \binom{n-4}{t-3} D_t \\ &\quad + 4 \binom{n-4}{r-3} \Big(\frac{2}{r-1} D_r - D_{r-2} - D_{r-3} \Big). \end{aligned} \tag{29}$$

From (23), (24), (25), (26), and (29) we finally have for $\alpha^{(1)}$:

$$|B_r(I_n) \cap B_r(\alpha^{(1)})| = \sum_{t=0}^{r-4} \bigg(\binom{n-4}{t} \big(D_t + 4D_{t+1} + 6D_{t+2} + 4D_{t+3} + D_{t+4} \big) \bigg)$$

$$+\binom{n-4}{r-3}\frac{4}{r-2}D_{r-1}+4\binom{n-4}{r-3}\Big(\frac{2}{r-1}D_r-D_{r-2}-D_{r-3}\Big). \tag{30}$$

Case 2: Let us compute $|B_r(I_n)\cap B_r(\alpha^{(2)})|$, where $\alpha^{(2)}$ is given by (19). Let $\sigma\in B_r(I_n)\cap B_r(\alpha^{(2)})$ and $d(\sigma, I_n)=t$. If $d(s_1,e_1)=0$ or $d(s_1,e_1)=1$ then $d(s_1,a_1^{(2)})=4$. Thus, if $d(s_1,e_1)=0$ then $t\leqslant r-4$; if $d(s_1,e_1)=1$ then $t\leqslant r-3$; if $d(s_1,e_1)=4$ then $t\leqslant r$. So if $d(s_1,e_1)=i$ for any $i\in\{0,1,4\}$ then the following equation holds:

$$|B_r(I_n)\cap B_r(\alpha^{(2)})|=|B_r(I_n)\cap B_r(\alpha^{(1)})|.$$

If $d(s_1,e_1)=3$ and $d(\sigma,I_n)\leqslant r-1$ then we also have Eq. (27). Consider $d(s_1,e_1)=3$ and $d(\sigma,I_n)=r$. Without loss of generality, $\sigma_1=1$. Then $d(\sigma,\alpha^{(2)})\leqslant r$ if and only if $\sigma_3=4$ or $\sigma_4=3$. By the above method, we also have the similar result:

$$\begin{aligned}&|\{\sigma\in B_r(I_n)\cap B_r(\alpha^{(2)})|\,\sigma_3=4 \text{ or } \sigma_4=3, \sigma_1=1, d(s_1,e_1)=3, d(\sigma,I_n)=r\}|\\&\qquad=\binom{n-4}{r-3}\Big(\frac{2}{r-1}D_r-D_{r-2}-D_{r-3}\Big).\end{aligned}$$

Therefore, if $d(s_1,e_1)=3$ then we also obtain that

$$|B_r(I_n)\cap B_r(\alpha^{(2)})|=|B_r(I_n)\cap B_r(\alpha^{(1)})|.$$

If $d(s_1,e_1)=2$ then $d(s_1,a_1^{(2)})\in\{2,3,4\}$. We choose $1,2$ from the set $\{1,2,3,4\}$ such that $\sigma_1\neq1$, $\sigma_2\neq2$. If $\sigma_1=2$ and $\sigma_2=1$ then $d(s_1,a_1^{(2)})=2$ and $d(\sigma,\alpha^{(2)})=t$. If $\sigma_1=2,\sigma_2\neq1$ or $\sigma_1\neq2,\sigma_2=1$ then $d(s_1,a_1^{(2)})=3$ and $d(\sigma,\alpha^{(2)})=t+1$. If $\sigma_1\neq2$ and $\sigma_2\neq1$ then $d(s_1,a_1^{(2)})=4$ and $d(\sigma,\alpha^{(2)})=t+2$. Hence, if $2\leqslant t\leqslant r-2$ then $\sigma\in B_r(I_n)\cap B_r(\alpha^{(2)})$, and the number of such permutations σ is $\sum\limits_{t=2}^{r-2}\binom{n-4}{t-2}D_t$.

If $t=r-1$ and $d(s_1,a_1^{(2)})\leqslant3$ then $\sigma\in B_r(I_n)\cap B_r(\alpha^{(2)})$. The number of the permutations σ in this case is $\binom{n-4}{r-3}\big(\frac{2}{r-2}D_{r-1}-D_{r-3}\big)$.

If $t=r$ and $d(s_1,a_1^{(2)})=2$ then $\sigma\in B_r(I_n)\cap B_r(\alpha^{(2)})$, and the number of these permutations σ is $\binom{n-4}{r-2}D_{r-2}$.

Therefore, if we choose $1,2$ then the number of sought permutations σ is totally given as follows:

$$\sum_{t=2}^{r-2}\binom{n-4}{t-2}D_t+\binom{n-4}{r-3}\Big(\frac{2}{r-2}D_{r-1}-D_{r-3}\Big)+\binom{n-4}{r-2}D_{r-2}. \tag{31}$$

Similarly, we get (31) if choosing $3,4$, and we have $d(s_1,a_1^{(2)})=4$ while choosing $1,3$; $1,4$; $2,3$; or $2,4$.

Thus, if $d(s_1,e_1)=2$ then by (31) totally we have:

$$\begin{aligned}|\{\sigma\in B_r(I_n)\cap B_r(\alpha^{(2)})|\,d(s_1,e_1)=2\}|&=6\sum_{t=2}^{r-2}\binom{n-4}{t-2}D_t+\frac{4}{r-2}\binom{n-4}{r-3}D_{r-1}\\&\quad+2\Big(\binom{n-4}{r-2}D_{r-2}-\binom{n-4}{r-3}D_{r-3}\Big).\end{aligned} \tag{32}$$

Therefore, by (23), (24), (25), (29), and (32) we have for $\alpha^{(2)}$:

$$|B_r(I_n) \cap B_r(\alpha^{(2)})| = \sum_{t=0}^{r-4} \left(\binom{n-4}{t} \Big(D_t + 4D_{t+1} + 6D_{t+2} + 4D_{t+3} + D_{t+4} \Big) \right) + \binom{n-4}{r-3} \frac{4}{r-2} D_{r-1} + 4\binom{n-4}{r-3} \Big(\frac{2}{r-1} D_r - D_{r-2} - D_{r-3} \Big) + 2\left(\binom{n-4}{r-2} D_{r-2} - \binom{n-4}{r-3} D_{r-3} \right). \quad (33)$$

Finally, by (30), (33) and by Lemma 5 we have (22) which completes the proof of Theorem 3. □

For any $n \geqslant 4$, the following particular cases are obtained from Lemma 8 and Theorem 3:

$$I(n,4,2) = 2, \quad (34)$$

$$I(n,4,3) = 10, \quad (35)$$

$$I(n,4,4) = n^2 + 15n - 52. \quad (36)$$

4 Exact values $N(n,r)$ for $r = 2, 3, 4$

In this section, we get the explicit formulas of $N(n,r)$ for $r = 2, 3, 4$.

4.1 The values of $N(n,2)$ and $N(n,3)$

Lemma 9 *$I(n,d,r) = 0$, where $2r + 1 \leqslant d \leqslant n$.*

Proof For any two distinct permutations $\pi, \sigma \in \mathrm{Sym}_n$ at distance $d(\pi,\sigma) = d$, if $d \geqslant 2r+1$ then $B_r(\pi) \cap B_r(\sigma) = \emptyset$. Hence, it follows that $I(n,d,r) = 0$ for $n \geqslant d \geqslant 2r+1$. □

If $r = 2$ then by (9), (15) and (34) we have $I(n,2,2) = 2$, $I(n,3,2) = 3$, $I(n,4,2) = 2$ and $I(n,d,2) = 0$ for any $d \geqslant 5$. Since $N(n,r) = \max_{d \geqslant 1} I(n,d,r)$ we immediately have the following result.

Theorem 4 *For any $n \geqslant 3$ we have*

$$N(n,2) = \max_{\pi,\sigma \in \mathrm{Sym}_n, \pi \neq \sigma} |B_2(\sigma) \cap B_2(\pi)| = 3.$$

Moreover, $N(2,2) = 2$.

Now we describe an algorithm for reconstruction of an unknown permutation $\pi \in \mathrm{Sym}_n$. Whenever a set $\{\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \pi^{(4)}\} \subseteq B_2(\pi)$ is known there are at most two errors on each channel (i.e., $r = 2$). Next, we give a revised majority algorithm on each bit of permutations to reconstruct the transmitted permutation in the following. Given a set $A = \{\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \pi^{(4)}\} \subseteq B_2(\pi)$ with $\pi \in \mathrm{Sym}_n$ and $n \geqslant 3$, we let the permutation $\pi^{(l)}$ for $l \in [4]$ as rows of a matrix $B = (b_{i,j})$ of size $4 \times n$, where $b_{i,j} = \pi^{(i)}(j)$ for all $i \in [4]$, $j \in [n]$. We use a revised majority algorithm to obtain $\hat{\pi}$ from A.

Revised Majority Algorithm: First, for each column $B_j = (b_{1,j}, b_{2,j}, b_{3,j}, b_{4,j})^T$ of B and $j \in [n]$, we apply the majority algorithm on B_j to get a value c_j. Specifically, if the value

c is the uniquely most frequently occurring component in B_j then we obtain that $c_j = c$; otherwise we let $c_j = \infty$. Therefore, we apply the majority algorithm on B to get a vector $(c_1, ..., c_n)$. Second, if $c_j \neq \infty$ then $\hat{\pi}(j) = c_j$; otherwise we let $\hat{\pi}(j) = \hat{c}$ such that $\hat{\pi} \in \mathrm{Sym}_n$.

Lemma 10 *Any $\pi \in \mathrm{Sym}_n$, $n \geqslant 3$, can be reconstructed by the Revised Majority Algorithm for a given $A = \{\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \pi^{(4)}\} \subseteq B_2(\pi)$.*

Proof Without loss of generality, let $\pi = I_n$. For any $l \in [4]$, since $\pi^{(l)} \in B_2(I_n)$ then we have $\pi^{(l)} = I_n$ or $\pi^{(l)} = (i\ j)$ with $i \neq j$, where $i, j \in [n]$. Next, we discuss the components of $B_k = (b_{1,k}, b_{2,k}, b_{3,k}, b_{4,k})^T$ for any $k \in [n]$. Assume we apply Revised Majority Algorithm on B_k to get some value c_k for any $k \in [n]$. Since $\pi^{(l)}$ is the identity permutation or a transposition, then $b_{l,k} = j$ and $j \neq k$ if and only if $\pi^{(l)} = (k\ j)$ for some $j, k \in [n]$ and $l \in [4]$. Thus, for any $j \neq k$ and $j \in [n]$, the components of B_k cannot have at least two j because $\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \pi^{(4)}$ are pairwise distinct. So, by using Revised Majority Algorithm on B_k we have $c_k = k$ or $c_k = \infty$ for any $k \in [n]$.

If there exist some $c_k = \infty$ then the components of B_k are all different. For convenience, let $b_{l,k} = j_l$ for $l \in [4]$. Next we discuss the two cases based on whether $k \in \{j_l | l \in [4]\}$ or not. When $k \notin \{j_l | l \in [4]\}$, we have $\pi^{(l)} = (k\ j_l)$ for $l \in [4]$. Thus, the components of B_i with $i \neq k$ have at least three i. So, by using Revised Majority Algorithm on B_i we have $c_i = i$ for any $i \in [n]$ and $i \neq k$. Therefore, we can determine the value of c_k because $[c_1\, c_2\, \ldots\, c_n]$ is a permutation. When $k \in \{j_l | l \in [4]\}$, assume that $j_1 = k$ then $\pi^{(1)} = I_n$ and $\pi^{(l)} = (k\ j_l)$ for $l \in \{2, 3, 4\}$. Thus, the components of B_i with $i \neq k$ have at least three i. Similarly, we also reconstruct I_n from A.

If there does not exist $c_k = \infty$ then we have $c_k = k$ for any $k \in [n]$. Therefore, we reconstruct I_n from A. □

In what follows below we give one example to reconstruct π from A by using Revised Majority Algorithm, where $\pi \in \mathrm{Sym}_n$, $A = \{\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \pi^{(4)}\} \subseteq B_2(\pi)$, and $n \geqslant 3$.

Example 1 Let $n = 5$ and $\pi = I_5 = [1\,2\,3\,4\,5]$. Given a set $A = \{\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \pi^{(4)}\} = \{[2\,1\,3\,4\,5], [1\,3\,2\,4\,5], [3\,2\,1\,4\,5], [1\,2\,3\,5\,4]\} \subseteq B_2(I_5)$ (in corresponding order), we use Revised Majority Algorithm to reconstruct I_5 from A. Consider the permutation $\pi^{(i)}$ for $i \in [4]$ as rows of a matrix B of size 4×5, then we have the following matrix B:

$$\begin{matrix} 2 & 1 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4. \end{matrix}$$

For each column B_i of B and $i \in [5]$, we use Revised Majority Algorithm to get the value (that is, i) from B_i for any $i \in [5]$. Therefore, we can reconstruct I_5 from A by using Revised Majority Algorithm.

If $r = 3$ then by (10), (16), and Lemma 8 we have $I(n, 2, 3) = 4n - 6$ and $I(n, 3, 3) = 3n - 3$ for any $n \geqslant 3$, $I(n, 4, 3) = 10$ for any $n \geqslant 4$, and $I(n, d, 3) = 0$ for any $d \geqslant 7$. By Theorems 1 and 2 from [24], we have the following results.

Lemma 11 $I(n, 5, 3) = 5$ *for any* $n \geqslant 5$, *and* $I(n, 6, 3) = 2$ *for any* $n \geqslant 6$.

Thus, finally for $r = 3$ we have the following theorem.

Theorem 5 *For any $n \geqslant 3$ we have*

$$N(n, 3) = \max_{\pi, \sigma \in \mathrm{Sym}_n, \pi \neq \sigma} |B_3(\pi) \cap B_3(\sigma)| = 4n - 6.$$

4.2 The value of $N(n, 4)$

If $r = 4$ then by (11), (17), (36) we have $I(n, 2, 4) = 7n^2 - 31n + 36$, $I(n, 3, 4) = \frac{9n^2-27n+12}{2}$, $I(n, 4, 4) = n^2 + 15n - 52$ for any $n \geqslant 4$, and $I(n, d, 4) = 0$ for any $d \geqslant 9$. In what follows below, we obtain $I(n, d, 4)$ for $d = 5, 6, 7, 8$. First, we determine $I(n, 5, 4)$ and $I(n, 6, 4)$.

Let $\pi \in \mathrm{Sym}_n$ be a product of disjoint cycles which we call cycles of π. We denote $Ts(\pi) = \{i \in [n] | \pi(i) \neq i\}$ and $Tc(\pi) = \{< i, \pi(i) > | \pi(i) \neq i \text{ for } i \in [n]\}$. For example, if $\pi = (1\,2)(4\,5\,6)$ then $Ts(\pi) = \{1, 2, 4, 5, 6\}$, $Tc(\pi) = \{< 1, 2 >, < 2, 1 >, < 4, 5 >, < 5, 6 >, < 6, 4 >\}$.

In order to get $I(n, d, 4)$ for $d = 5, 6$ we need additional results. The next lemma describes $disc(\pi)$ for any $\pi \in \mathrm{Sym}_n$ with $d(I_n, \pi) = 5, 6$.

Lemma 12 *Let $\pi \in \mathrm{Sym}_n$. Then we have:*

$$disc(\pi) = \begin{cases} [1^{n-5}2^1 3^1], \ or \ [1^{n-5}5^1] & if \ d(I_n, \pi) = 5, \\ [1^{n-6}2^3], [1^{n-6}3^2], [1^{n-6}2^1 4^1], \ or \ [1^{n-6}6^1] & if \ d(I_n, \pi) = 6. \end{cases} \tag{37}$$

Proof Let $disc(\pi) = [1^{h_1}2^{h_2} \ldots n^{h_n}]$ with $h_i \geqslant 0$ for any $i \in [n]$. We easily obtain that $d(I_n, \pi) = \sum_{i=2}^{n} i h_i$. If $d(I_n, \pi) = 5$ then $\sum_{i=2}^{n} i h_i = 5$ which gives us the following two cases: either $h_1 = n - 5, h_2 = 1, h_3 = 1$ or $h_1 = n - 5, h_5 = 1$. Hence, $disc(\pi)$ is equal either $[1^{n-5}2^1 3^1]$ or $[1^{n-5}5^1]$ in this case. Similarly, we also obtain the above results of $disc(\pi)$ for $d(I_n, \pi) = 6$. □

For any two permutations $\pi, \tau \in \mathrm{Sym}_n$, it is easily verified that the following equality holds:

$$d(\pi, \tau) = |Ts(\pi) \cup Ts(\tau)| - |Tc(\pi) \cap Tc(\tau)|. \tag{38}$$

Lemma 13 $I(n, 5, 4) = 6n + 14$ *for any $n \geqslant 6$, where* $I(5, 5, 4) = 45$.

Proof By Lemmas 2 and 12, we have:

$$I(n, 5, 4) = \max\{|B_4(\pi) \cap B_4(I_n)|, |B_4(\tau) \cap B_4(I_n)|\}, \tag{39}$$

where $d(I_n, \pi) = d(I_n, \tau) = 5$, $\pi = (1\,2)(3\,4\,5)$, and $\tau = (1\,2\,3\,4\,5)$.

First, we find $|B_4(\pi) \cap B_4(I_n)|$. For any $\sigma \in B_4(\pi) \cap B_4(I_n)$, we have $d(I_n, \sigma) \leqslant 4$ and $d(\sigma, \pi) \leqslant 4$ which immediately gives us:

$$|B_4(\pi) \cap B_4(I_n)| = \sum_{y=0}^{4} |\{\sigma | d(I_n, \sigma) = y, d(\sigma, \pi) \leqslant 4\}|.$$

It is clear that if $y = 0$ or $y = 1$ then $|\{\sigma | d(I_n, \sigma) = y, d(\sigma, \pi) \leqslant 4\}| = 0$.

If $d(I_n, \sigma) = 2$ then σ is given by one of the transpositions $(i\ j)$ with $i \neq j$, where $i, j \in [n]$. Since $Tc(\pi) = \{<1,2>, <2,1>, <3,4>, <4,5>, <5,3>\}$ then by (38) we have:

$$d(\sigma,\pi) = |[5] \cup \{i, j\}| - \\ |\{<i, j>, <j, i>\} \cap \{<1,2>, <2,1>, <3,4>, <4,5>, <5,3>\}|$$

Thus, if $d(\sigma, \pi) \leqslant 4$ with $\sigma = (i\ j)$ then σ can be represented by $(1\,2)$, $(3\,4)$, $(3\,5)$, or $(4\,5)$. Therefore, we have:

$$|\{\sigma | d(I_n, \sigma) = 2, d(\sigma, \pi) \leqslant 4\}| = 4.$$

If $d(I_n, \sigma) = 3$ then $\sigma = (i\,j\,k)$ with $i \neq j \neq k$, where $i, j, k \in [n]$. If there exist at least one element from the set $\{i, j, k\}$ which are not in [5], by (38) we have $d(\sigma, \pi) \geqslant 5$. Thus, we have $i, j, k \in [5]$. To satisfy the condition of $d(\sigma, \pi) \leqslant 4$, by (38) we have $\{<i, j>, <j, k>, <k, i>\} \cap \{<1,2>, <2,1>, <3,4>, <4,5>, <5,3>\} \neq \emptyset$. Thus, we have the following five cases of σ: 1) $\sigma = (1\,2\,t)$ or $\sigma = (2\,1\,t)$ for $t \in \{3, 4, 5\}$; 2) $\sigma = (3\,4\,t)$ for $t \in \{1, 2, 5\}$; 3) $\sigma = (4\,5\,t)$ for $t \in [3]$; 4) $\sigma = (5\,3\,t)$ for $t \in \{1, 2, 4\}$. Since $(3\,4\,5)$ is counted in the cases of 2), 3) and 4), hence we have:

$$|\{\sigma | d(I_n, \sigma) = 3, d(\sigma, \pi) \leqslant 4\}| = 3 \times 5 - 2 = 13.$$

If $d(I_n, \sigma) = 4$ then either $\sigma = (i\ j)(k\,l)$ or $\sigma = (i\,j\,k\,l)$ with $i \neq j \neq k \neq l$, where $i, j, k, l \in [n]$. Let us consider $\sigma = (i\ j)(k\,l)$. If there exist at least two elements from the set $\{i, j, k, l\}$ which are not in [5], by (38) we have $d(\sigma, \pi) \geqslant 5$. Thus there exists at most one element from $\{i, j, k, l\}$ which is not in [5]. Without loss of generality, we let $i, j, k \in [5]$ and $l \notin [5]$. If $(i\ j) \neq (1\,2)$ then $d(\sigma, \pi) \geqslant 5$. If $(i\ j) = (1\,2)$ then $d(\sigma, \pi) = 4$, where $k \in \{3, 4, 5\}$ and $l \in [n]\backslash[5]$. Thus, we have:

$$|\{\sigma | \sigma = (i\ j)(k\,l),\ i, j, k \in [5], l \notin [5], d(I_n, \sigma) = 4, d(\sigma, \pi) \leqslant 4\}| = 3n - 15.$$

When $i, j, k, l \in [5]$, to satisfy the condition of $d(\sigma, \pi) \leqslant 4$, then by (38) we have $\{<i, j>, <j, i>, <k, l>, <l, k>\} \cap \{<1,2>, <2,1>, <3,4>, <4,5>, <5,3>\} \neq \emptyset$. Thus, σ is represented by one of the following four permutations: 1) $\sigma = (1\,2)(k\,l)$ for $k, l \in \{3, 4, 5\}$; 2) $\sigma = (t\,3)(4\,5)$ for $t \in [2]$; 3) $\sigma = (t\,4)(3\,5)$ for $t \in [2]$; 4) $\sigma = (t\,5)(3\,4)$ for $t \in [2]$, and in this case we have:

$$|\{\sigma | \sigma = (i\ j)(k\,l),\ i, j, k, l \in [5], d(I_n, \sigma) = 4, d(\sigma, \pi) \leqslant 4\}| = 9.$$

Now let us consider $\sigma = (i\,j\,k\,l)$. There exists at most one element from the set $\{i, j, k, l\}$ which is not in [5]. Without loss of generality, we let $i, j, k \in [5]$. When $l \notin [5]$, to make $d(\sigma, \pi) \leqslant 4$, then by (38) we have $|\{<i, j>, <j, k>, <k, l>, <l, i>\} \cap \{<1,2>, <2,1>, <3,4>, <4,5>, <5,3>\}| \geqslant 2$. Thus, we have $\sigma = (3\,4\,5\,l), (4\,5\,3\,l)$, or $(5\,3\,4\,l)$ such that $d(\sigma, \pi) = 4$ for any $l \in [n]\backslash[5]$. When $l \in [5]$, to satisfy the condition of $d(\sigma, \pi) \leqslant 4$, then by (38) we have $\{<i, j>, <j, k>, <k, l>, <l, i>\} \cap \{<1,2>, <2,1>, <3,4>, <4,5>, <5,3>\} \neq \emptyset$. Thus, we have 1) $\sigma = (1\,2\,k\,l)$ or $\sigma = (2\,1\,k\,l)$ for any $k, l \in \{3, 4, 5\}$; 2) $\sigma = (3\,4\,1\,5)$, or $\sigma = (3\,4\,5\,1)$, or $\sigma = (3\,4\,2\,5)$, or $\sigma = (3\,4\,5\,2)$, or $\sigma = (4\,5\,3\,1)$, or $\sigma = (4\,5\,3\,2)$. Therefore, we have:

$$|\{\sigma | \sigma = (i\,j\,k\,l),\ i, j, k, l \in [n], d(I_n, \sigma) = 4, d(\sigma, \pi) \leqslant 4\}| = 3(n-5) + 18 = 3n + 3.$$

So, if $d(I_n, \sigma) = 4$ then totally we have:

$$|\{\sigma | d(I_n, \sigma) = 4, d(\sigma, \pi) \leqslant 4\}| = 3n - 15 + 9 + 3n + 3 = 6n - 3,$$

and for $\pi = (1\,2)(3\,4\,5)$ finally we have:

$$|B_4(\pi) \cap B_4(I_n)| = 4 + 13 + 6n - 3 = 6n + 14. \tag{40}$$

Now we consider $\tau = (1\,2\,3\,4\,5)$ and compute $|B_4(\tau) \cap B_4(I_n)|$, where

$$|B_4(\tau) \cap B_4(I_n)| = \sum_{y=0}^{4} |\{\sigma | d(I_n, \sigma) = y, d(\sigma, \tau) \leqslant 4\}|.$$

It is clear that if $y = 0$ or $y = 1$ then $|\{\sigma | d(I_n, \sigma) = y, d(\sigma, \tau) \leqslant 4\}| = 0$.

When $d(I_n, \sigma) = 2$ then by (38) a permutation σ is represented by one of the transpositions from the set $\{(1\,2), (2\,3), (3\,4), (4\,5), (5\,1)\}$. Therefore, we have:

$$|\{\sigma | d(I_n, \sigma) = 2, d(\sigma, \tau) \leqslant 4\}| = 5.$$

When $d(I_n, \sigma) = 3$, let $\sigma = (i\ j\ k)$ for $i, j, k \in [n]$. If there exists one element from the set $\{i, j, k\}$ which is not in [5] then by (38) $d(\sigma, \tau) \geqslant 5$. Thus, we have $i, j, k \in [5]$. So, σ is given by one of the permutations from the set

$$\{(1\,2\,3), (1\,2\,4), (1\,2\,5), (1\,3\,4), (1\,3\,5), (1\,4\,5), (2\,3\,4), (2\,3\,5), (2\,4\,5), (3\,4\,5)\}.$$

Therefore, we have:

$$|\{\sigma | d(I_n, \sigma) = 3, d(\sigma, \tau) \leqslant 4\}| = 10.$$

When $d(I_n, \sigma) = 4$ then $\sigma = (i\ j)(k\,l)$ or $\sigma = (i\ j\ k\,l)$ with $i \neq j \neq k \neq l$, where $i, j, k, l \in [n]$. If $\sigma = (i\ j)(k\,l)$ then by (38) we have $i, j, k, l \in [5]$ such that $d(\sigma, \tau) \leqslant 4$. Thus, σ can be represented by one of the following permutations: 1) $(1\,2)(k\,l)$, where $k, l \in \{3, 4, 5\}$; 2) $(2\,3)(k\,l)$, where $k, l \in \{1, 4, 5\}$; 3) $(3\,4)(1\,5)$, $(3\,4)(2\,5)$, $(4\,5)(1\,3)$, or $(5\,1)(2\,4)$. Thus, we have:

$$|\{\sigma | \sigma = (i\ j)(k\,l), i, j, k, l \in [n], d(I_n, \sigma) = 4, d(\sigma, \tau) \leqslant 4\}| = 10.$$

If $\sigma = (i\ j\ k\,l)$ then by (38) there exists at most one element from $\{i, j, k, l\}$ which is not in [5]. When $i, j, k, l \in [5]$, without loss of generality, we let $i, j, k, l \in [4]$. Then $\{< i, j >, < j, k >, < k, l >, < l, i >\} \cap \{< 1, 2 >, < 2, 3 >, < 3, 4 >, < 4, 5 >, < 5, 1 >\} \neq \emptyset$. Thus, $\sigma \in \{(1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,4\,2), (1\,4\,2\,3)\}$, where $i, j, k, l \in [4]$, and we have:

$$|\{\sigma | \sigma = (i\ j\ k\,l), i, j, k, l \in [5], d(I_n, \sigma) = 4, d(\sigma, \tau) \leqslant 4\}| = 4 \times 5 = 20.$$

If $i, j, k \in [5]$ and $l \notin [5]$ then $|\{< i, j >, < j, k >, < k, l >, < l, i >\} \cap \{< 1, 2 >, < 2, 3 >, < 3, 4 >, < 4, 5 >, < 5, 1 >\}| \geqslant 2$ such that $d(\sigma, \tau) \leqslant 4$. Thus, $\sigma \in \{(1\,2\,3\,t), (2\,3\,4\,t), (3\,4\,5\,t), (4\,5\,1\,t), (5\,1\,2\,t)\}$, where $t \in [n] \backslash [5]$, and we have:

$$|\{\sigma | \sigma = (i\ j\ k\,l), i, j, k \in [5], l \notin [5], d(I_n, \sigma) = 4, d(\sigma, \tau) \leqslant 4\}| = 5n - 25,$$

and totally for $d(I_n, \sigma) = 4$ we have:

$$|\{\sigma | d(I_n, \sigma) = 4, d(\sigma, \tau) \leqslant 4\}| = 5n - 25 + 20 + 10 = 5n + 5,$$

which together with results for $d(I_n, \sigma) = 2$ and $d(I_n, \sigma) = 3$ gives:

$$|B_4(\tau) \cap B_4(I_n)| = 5 + 10 + 5n + 5 = 5n + 20, \tag{41}$$

where $\tau = (1\,2\,3\,4\,5)$. Therefore, by (39)–(41), we have $I(n, 5, 4) = 6n + 14$ for any $n \geqslant 6$ and $I(5, 5, 4) = 45$. □

Lemma 14 $I(n, 6, 4) = 38$ *for any* $n \geqslant 6$.

Proof By Lemma 12 and (38), we have:

$$I(n, 6, 4) = \max\{|B_4(\pi_x) \cap B_4(I_n)|, 1 \leqslant x \leqslant 4\},$$

where $d(I_n, \pi_x) = 6$ for any $x \in [4]$ with $\pi_1 = (1\,2)(3\,4)(5\,6)$, $\pi_2 = (1\,2\,3)(4\,5\,6)$, $\pi_3 = (1\,2)(3\,4\,5\,6)$, and $\pi_4 = (1\,2\,3\,4\,5\,6)$. Moreover, for any $x \in [4]$ we have:

$$|B_4(\pi_x) \cap B_4(I_n)| = \sum_{y=0}^{4} |\{\sigma | d(I_n, \sigma) = y, d(\sigma, \pi_x) \leqslant 4\}|.$$

Now let us check all possible cases.

If $y = 0$ or $y = 1$ then $|\{\sigma | d(I_n, \sigma) = y, d(\sigma, \pi_x) \leqslant 4\}| = 0$ for any $x \in [4]$.

If $d(I_n, \sigma) = 2$ or $d(I_n, \sigma) = 3$ then a permutation σ is represented as a transposition $\sigma = (i\ j)$ or as a 3-cycle $\sigma = (i\ j\ k)$. If $d(\sigma, \pi_x) \leqslant 4$ for any $x \in [4]$ then by (38) we have $i, j, k \in [6]$.

If $d(I_n, \sigma) = 4$ then there are two possibilities for σ to be presented as a composition of two 2-cycles $\sigma = (i\ j)(k\,l)$ or as a 4-cycle $\sigma = (i\ j\ k\,l)$. If $d(\sigma, \pi_x) \leqslant 4$ for any $x \in [4]$ then $i, j, k, l \in [6]$.

First, let us find $|B_4(\pi_1) \cap B_4(I_n)|$ for $\pi_1 = (1\,2)(3\,4)(5\,6)$. Let $\sigma \in B_4(\pi_1) \cap B_4(I_n)$. If $d(I_n, \sigma) = 2$ then $\sigma = (i\ j), i, j \in [6]$. Since

$$Tc(\pi_1) = \{< 1, 2 >, < 2, 1 >, < 3, 4 >, < 4, 3 >, < 5, 6 >, < 6, 5 >\}$$

then by (38) a permutation σ can be represented by one of the transpositions $(1\,2)$, $(3\,4)$, or $(5\,6)$ such that $d(\sigma, \pi_1) \leqslant 4$. Thus, we have:

$$|\{\sigma | d(I_n, \sigma) = 2, d(\sigma, \pi_1) \leqslant 4\}| = 3.$$

If $d(I_n, \sigma) = 3$ then $\sigma = (i\ j\ k)$, where $i, j, k \in [6]$. By (38) there does not exist a permutation $\sigma \in \mathrm{Sym}_n$ with $d(I_n, \sigma) = 3$ such that $d(\sigma, \pi_1) \leqslant 4$, hence, we have:

$$|\{\sigma | d(I_n, \sigma) = 3, d(\sigma, \pi_1) \leqslant 4\}| = 0.$$

If $d(I_n, \sigma) = 4$ then either $\sigma = (i\ j)(k\,l)$ or $\sigma = (i\ j\ k\,l)$, where $i, j, k, l \in [6]$. When $\sigma = (i\ j)(k\,l)$, by (38) we have $|\{< i, j >, < j, i >, < k, l >, < l, k >\} \cap Tc(\pi_1)| \geqslant 2$ such that $d(\sigma, \pi_1) \leqslant 4$. Thus, σ can be represented by one of the following permutations: 1) $(1\,2)(k\,l)$, where $k, l \in \{3, 4, 5, 6\}$; 2) $(3\,4)(k\,l)$, where $k, l \in \{1, 2, 5, 6\}$; 3) $(5\,6)(k\,l)$, where $k, l \in [4]$. Since $(1\,2)(3\,4)$, $(1\,2)(5\,6)$, and $(3\,4)(5\,6)$ appear twice, the total number of permutations $\sigma = (i\ j)(k\,l)$ is given by $3 \cdot \binom{4}{2} - 3$. If $\sigma = (i\ j\ k\,l)$ and $d(\sigma, \pi_1) \leqslant 4$ then by (38) we have $|\{< i, j >, < j, k >, < k, l >, < l, i >\} \cap Tc(\pi_1)| \geqslant 2$. Since $Tc(\pi_1) = \{< 1, 2 >, < 2, 1 >, < 3, 4 >, < 4, 3 >, < 5, 6 >, < 6, 5 >\}$, and $< a, b >, < b, a >$ are not both in $Tc(\sigma)$ simultaneously for some $\{a, b\} = \{1, 2\}, \{3, 4\}$, or $\{5, 6\}$, we must choose i, j, k, l from the sets $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$, or $\{3, 4, 5, 6\}$ such that $|Tc(\sigma) \cap Tc(\pi_1)| \geqslant 2$. For each of these cases, we have one and the same set of permutations as follows: $\{(1\,2\,3\,4), (1\,2\,4\,3), (2\,1\,3\,4), (2\,1\,4\,3)\}$. Thus, the total number of permutations $\sigma = (i\ j\ k\,l)$ is equal to 12, and we have:

$$|\{\sigma | d(I_n, \sigma) = 4, d(\sigma, \pi_1) \leqslant 4\}| = 3 \cdot \binom{4}{2} - 3 + 12 = 27.$$

Finally, for $\pi_1 = (1\,2)(3\,4)(5\,6)$ we have:

$$|B_4(\pi_1) \cap B_4(I_n)| = 3 + 27 = 30. \tag{42}$$

Now let us find $|B_4(\pi_2) \cap B_4(I_n)|$ for $\pi_2 = (1\,2\,3)(4\,5\,6)$. Let $\sigma \in B_4(\pi_2) \cap B_4(I_n)$. If $d(I_n, \sigma) = 2$ then by (38) we have $d(\sigma, \pi_2) = |Ts(\sigma) \cup Ts(\pi_2)| - |Tc(\sigma) \cap Tc(\pi_2)| \geqslant 6 - 1 = 5$, where $Tc(\pi_2) = \{<1, 2>, <2, 3>, <3, 1>, <4, 5>, <5, 6>, <6, 4>\}$. Thus, we have:

$$|\{\sigma | d(I_n, \sigma) = 2, d(\sigma, \pi_2) \leqslant 4\}| = 0.$$

If $d(I_n, \sigma) = 3$ then since σ can be represented by either $(1\,2\,3)$ or $(4\,5\,6)$, hence, we have:

$$|\{\sigma | d(I_n, \sigma) = 3, d(\sigma, \pi_2) \leqslant 4\}| = 2.$$

Let $d(I_n, \sigma) = 4$. Then $\sigma = (i\,j)(k\,l)$ or $(i\,j\,k\,l)$ for $i, j, k, l \in [n]$. Moreover, by (38) we have $|Tc(\sigma) \cap Tc(\pi_2)| \geqslant 2$ and $i, j, k, l \in [6]$ such that $d(\sigma, \pi_2) \leqslant 4$. If $\sigma = (i\,j)(k\,l)$ with $Tc(\sigma) = \{<i, j>, <j, i>, <k, l>, <l, k>\}$ then by (38) we have $(i\,j) \in \{(1\,2), (2\,3), (3\,1)\}$ and $(k\,l) \in \{(4\,5), (5\,6), (6\,4)\}$. If $\sigma = (i\,j\,k\,l)$ then there are a few possibilities for σ: either $\sigma \in \{(1\,2\,3\,l), (2\,3\,1\,l), (3\,1\,2\,l)\}$, where $l \in \{4, 5, 6\}$, or $\sigma \in \{(4\,5\,6\,l), (5\,6\,4\,l), (6\,4\,5\,l)\}$, where $l \in [3]$, or $\sigma = (i\,j\,k\,l)$, where $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$ and $(k, l) \in \{(4, 5), (5, 6), (6, 4)\}$. Thus, in this case we have:

$$|\{\sigma | d(I_n, \sigma) = 4, d(\sigma, \pi_2) \leqslant 4\}| = 36,$$

and finally for $\pi_2 = (1\,2\,3)(4\,5\,6)$ we have:

$$|B_4(\pi_2) \cap B_4(I_n)| = 2 + 36 = 38. \tag{43}$$

Now let us find $|B_4(\pi_3) \cap B_4(I_n)|$, where $\pi_3 = (1\,2)(3\,4\,5\,6)$.

Let $\sigma \in B_4(\pi_3) \cap B_4(I_n)$. If $d(I_n, \sigma) = 2$ then since

$$Tc(\pi_3) = \{<1, 2>, <2, 1>, <3, 4>, <4, 5>, <5, 6>, <6, 3>\}$$

by (38) we have $\sigma = (1\,2)$ which gives us:

$$|\{\sigma | d(I_n, \sigma) = 2, d(\sigma, \pi_3) \leqslant 4\}| = 1.$$

If $d(I_n, \sigma) = 3$ then $\sigma = (i\,j\,k)$, where $i, j, k \in [6]$. By (38) we have $|Tc(\sigma) \cap Tc(\pi_3)| \geqslant 2$ such that $d(\sigma, \pi_3) \leqslant 4$. Since $Tc(\sigma) = \{<i, j>, <j, k>, <k, i>\}$ then $<1, 2>$ and $<2, 1>$ are not in $Tc(\sigma)$. Moreover, any two distinct elements in $Tc(\sigma)$ can determine σ. Thus, $\sigma \in \{(3\,4\,5), (4\,5\,6), (5\,6\,3), (6\,3\,4)\}$ which gives:

$$|\{\sigma | d(I_n, \sigma) = 3, d(\sigma, \pi_3) \leqslant 4\}| = 4.$$

If $d(I_n, \sigma) = 4$ then by (38) we have $\sigma = (i\,j)(k\,l)$ or $(i\,j\,k\,l)$ for $i, j, k, l \in [6]$. Moreover, we have $|Tc(\sigma) \cap Tc(\pi_3)| \geqslant 2$ such that $d(\sigma, \pi_3) \leqslant 4$. First, we let $\sigma = (i\,j)(k\,l)$. When $(i\,j) = (1\,2)$, it is easily verified that $d(\sigma, \pi_3) \leqslant 4$ for any $k, l \in \{3, 4, 5, 6\}$. When $(i\,j) \in \{(1\,j), (2\,j) | j \in \{3, 4, 5, 6\}\}$, then $|Tc(\sigma) \cap Tc(\pi_3)| \leqslant 1$ such that $d(\sigma, \pi_3) \geqslant 5$. Thus, σ can be given by one of the following permutations: 1) $(1\,2)(k\,l)$, where $k, l \in \{3, 4, 5, 6\}$; 2) $(3\,4)(5\,6)$ or $(4\,5)(6\,3)$. Second, we let $\sigma = (i\,j\,k\,l)$. When $<1, 2>$ or $<2, 1>$, and $<k, l> \in Tc(\sigma) \cap Tc(\pi_3)$ for some $k, l \in \{3, 4, 5, 6\}$ then $(k, l) \in \{(3, 4), (4, 5), (5, 6), (6, 3)\}$ such that $d(\sigma, \pi_3) \leqslant 4$. When $<1, 2>$ and $<2, 1>$ do not belong to $Tc(\sigma)$, we can choose two elements from $\{(3, 4), (4, 5), (5, 6), (6, 3)\}$ to construct σ such that $\sigma \in \{(3\,4\,5\,l), (4\,5\,6\,l), (5\,6\,3\,l), (6\,3\,4\,l), (3\,4\,5\,6) | l \in [2]\}$. Thus, $\sigma = (i\,j\,k\,l)$ can be represented by one of the following permutations: 1) either $(1\,2\,k\,l)$ or $(2\,1\,k\,l)$, where $(k, l) \in \{(3, 4), (4, 5), (5, 6), (6, 3)\}$; 2) either $(3\,4\,5\,l)$ or $(4\,5\,6\,l)$ or $(5\,6\,3\,l)$ or $(6\,3\,4\,l)$, where $l \in [2]$; 3) $(3\,4\,5\,6)$ which in this case gives us:

$$|\{\sigma | d(I_n, \sigma) = 4, d(\sigma, \pi_3) \leqslant 4\}| = 8 + 8 + 8 + 1 = 25,$$

and totally for $\pi_3 = (1\,2)(3\,4\,5\,6)$ we have:

$$|B_4(\pi_3) \cap B_4(I_n)| = 1 + 4 + 25 = 30. \tag{44}$$

Finally, we find $|B_4(\pi_4) \cap B_4(I_n)|$, where $\pi_4 = (1\,2\,3\,4\,5\,6)$.
Let $\sigma \in B_4(\pi_4) \cap B_4(I_n)$. If $d(I_n, \sigma) = 2$ then since

$$Tc(\pi_4) = \{< 1, 2 >, < 2, 3 >, < 3, 4 >, < 4, 5 >, < 5, 6 >, < 6, 1 >\}$$

by (38) we have:

$$|\{\sigma | d(I_n, \sigma) = 2, d(\sigma, \pi_4) \leqslant 4\}| = 0.$$

If $d(I_n, \sigma) = 4$ then by (38) we have $\sigma = (i\,j)(k\,l)$ or $\sigma = (i\,j\,k\,l)$, where $i, j, k, l \in [6]$. When $\sigma = (i\,j)(k\,l)$, we can choose two elements in $Tc(\pi_4)$ to construct σ, where $i \neq j \neq k \neq l$. Thus, σ should belong to the set $\{(1\,2)(3\,4), (1\,2)(4\,5), (1\,2)(5\,6), (2\,3)(4\,5), (2\,3)(5\,6), (2\,3)(6\,1), (3\,4)(5\,6), (3\,4)(6\,1), (4\,5)(6\,1)\}$. When $\sigma = (i\,j\,k\,l)$, then there may exist $< i, j >, < j, k > \in Tc(\pi_4)$, or $< i, j >, < k, l > \in Tc(\pi_4)$ such that $d(\sigma, \pi_4) \leqslant 4$. If $< i, j >$ and $< j, k > \in Tc(\pi_4)$ then $\sigma = (1\,2\,3\,l), (2\,3\,4\,l), (3\,4\,5\,l), (4\,5\,6\,l), (5\,6\,1\,l), (6\,1\,2\,l)$ for some $l \in [6]$. If $< i, j >$, $< k, l > \in Tc(\pi_4)$, and there does not exist three consecutive elements in $\{i, j, k, l\}$ then $\sigma =(1\,2\,4\,5),(2\,3\,5\,6),(3\,4\,6\,1)$. Thus, if $\sigma = (i\,j\,k\,l)$ then there are the following possibilities: 1) $(1\,2\,3\,l)$, $l \in \{4, 5, 6\}$; 2) $(2\,3\,4\,l)$, $l \in \{5, 6\}$; 3) $(3\,4\,5\,l)$, $l \in \{1, 6\}$; 4) $(4\,5\,6\,l)$, $l \in \{1, 2\}$; 5) $(5\,6\,1\,l)$, $l \in \{2, 3\}$; 6) $(6\,1\,2\,4)$,$(1\,2\,4\,5)$,$(2\,3\,5\,6)$,$(3\,4\,6\,1)$ which gives us:

$$|\{\sigma | d(I_n, \sigma) = 4, d(\sigma, \pi_4) \leqslant 4\}| = 9 + 15 = 24,$$

and totally for $\pi_4 = (1\,2\,3\,4\,5\,6)$ we have:

$$|B_4(\pi_4) \cap B_4(I_n)| = 6 + 24 = 30. \tag{45}$$

Taking into account (42)–(45), for any $n \geqslant 6$ we finally obtain:

$$I(n, 6, 4) = \max\{|B_4(\pi_x) \cap B_4(I_n)|, 1 \leqslant x \leqslant 4\} = 38$$

which completes the proof. □

The following result is a corollary from Theorem 1 and Example 2 in [24].

Lemma 15 *For any $n \geqslant d$, we have:*

$$I(n, d, 4) = \begin{cases} 10 & \textit{if } d = 7, \\ 6 & \textit{if } d = 8. \end{cases}$$

Theorem 6 *For any $n \geqslant 4$ we have:*

$$N(n, 4) = \max_{\pi, \sigma \in \mathrm{Sym}_n, \pi \neq \sigma} |B_4(\pi) \cap B_4(\sigma)| = 7n^2 - 31n + 36.$$

Proof The result immediately follows from (11), (17), (36), and Lemmas 13, 15. □

We say that a permutation $\pi \in \mathrm{Sym}_n$ is *reconstructible* from distinct permutations $\sigma_1, \sigma_2, \ldots, \sigma_h \in B_r(\pi)$ if there does not exist a permutation $\tau \in \mathrm{Sym}_n$, $\tau \neq \pi$, such that $\sigma_1, \sigma_2, \ldots, \sigma_h \in B_r(\tau)$. It is obvious, that if $r = n$ then for any two distinct permutations $\pi, \tau \in \mathrm{Sym}_n$ their metric balls $B_r(\pi)$ and $B_r(\tau)$ coincide with the symmetric group Sym_n. So, any permutation π is not reconstructible from distinct permutations $\sigma_1, \sigma_2, \ldots, \sigma_h \in B_n(\pi)$. From this definition and by Theorems 4, 5, 6 we have the following result.

Theorem 7 *For any permutation $\pi \in \mathrm{Sym}_n$, the following holds:*

1. *π is reconstructible from any 4 distinct permutations in $B_2(\pi)$ for any $n \geqslant 3$;*
2. *π is reconstructible from any $4n-5$ distinct permutations in $B_3(\pi)$ for any $n \geqslant 4$;*
3. *π is reconstructible from any $7n^2-31n+37$ distinct permutations in $B_4(\pi)$ for any $n \geqslant 5$.*

Proof By Theorem 4 we have $N(n,2)=3$. Moreover, by (4) we have $B_2(n)=1+\binom{n}{2}=\frac{n^2-n+2}{2}$ for any $n \geqslant 2$. This gives us $N(n,2) \leqslant B_2(n)-1$ for any $n \geqslant 3$. Hence, for any $n \geqslant 3$ any permutation $\pi \in \mathrm{Sym}_n$ is reconstructible from any four distinct permutations in $B_2(\pi)$.

If $r=3$ then by Theorem 5 we have $N(n,3)=4n-6$, and by (4) we have $B_3(n)=1+\binom{n}{2}+2\binom{n}{3}=\frac{2n^3-3n^2+n+6}{6}$ for any $n \geqslant 3$. Thus, $N(n,3) \leqslant B_3(n)-1$ for any $n \geqslant 4$ which means that for any $n \geqslant 4$ any permutation $\pi \in \mathrm{Sym}_n$ is reconstructible from any $4n-5$ distinct permutations in $B_3(\pi)$.

In a similar way it is shown that any permutation π is reconstructible from any $7n^2-31n+37$ distinct permutations in $B_4(\pi)$ for any $n \geqslant 5$. □

5 An asymptotic behaviour of $N(n, r)$

In this section we study $N(n,r)$ for any $r \geqslant 2$ and for sufficiently large n. Moreover, for $r=2$ we obtain a probability of the event that a permutation $\pi \in \mathrm{Sym}_n$ is reconstructible from at most four distinct permutations in $B_2(\pi)$.

First, we get a lower bound on $N(n,r)$.

Theorem 8 *For $r \geqslant 5$, the following holds:*

$$N(n,r) \geqslant \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2}\big). \tag{46}$$

Proof By Theorems 1, 2, 3 we have $I(n,d,r)$ for $d=2,3,4$ and for any $r \geqslant 5$. Now let us compare $I(n,2,r)$, $I(n,3,r)$, and $I(n,4,r)$.

Let us prove that $I(n,2,r) > I(n,3,r)$ for any $r \geqslant 5$. Indeed, from Theorems 1 and 2 it follows that we have:

$$I(n,2,r) = \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2}\big)$$

and

$$I(n,3,r) = \sum_{t=0}^{r-3} \left(\binom{n-3}{t} \big(D_t + 3D_{t+1} + 3D_{t+2} + D_{t+3}\big) \right) + \frac{3}{r-1} \binom{n-3}{r-2} D_r.$$

Since $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ then for any $r \geqslant 5$ we have:

$$I(n,2,r) = \sum_{t=0}^{r-2} \left(\binom{n-3}{t} + \binom{n-3}{t-1} \right) \Big(D_t + 2D_{t+1} + D_{t+2} \Big)$$

$$
\begin{aligned}
&= \sum_{t=0}^{r-2} \binom{n-3}{t} \Big(D_t + 2D_{t+1} + D_{t+2} \Big) + \sum_{t=0}^{r-2} \binom{n-3}{t-1} \Big(D_t + 2D_{t+1} + D_{t+2} \Big) \\
&= \sum_{t=0}^{r-3} \binom{n-3}{t} \Big(D_t + 3D_{t+1} + 3D_{t+2} + D_{t+3} \Big) + \binom{n-3}{r-2} \Big(D_{r-2} + 2D_{r-1} + D_r \Big) \\
&> \sum_{t=0}^{r-3} \binom{n-3}{t} \Big(D_t + 3D_{t+1} + 3D_{t+2} + D_{t+3} \Big) + \frac{3}{r-1} \binom{n-3}{r-2} D_r = I(n,3,r).
\end{aligned}
$$

Now we prove that $I(n,2,r) > I(n,4,r)$ for any $r \geqslant 5$. By Theorem 3, we have:

$$
I(n,4,r) = \sum_{t=0}^{r-4} \left(\binom{n-4}{t} (D_t + 4D_{t+1} + 6D_{t+2} + 4D_{t+3} + D_{t+4}) \right) + \Delta,
$$

where

$$
\begin{aligned}
\Delta =& 2 \cdot \max \left\{ \left(\binom{n-4}{r-2} D_{r-2} - \binom{n-4}{r-3} D_{r-3} \right), 0 \right\} + \binom{n-4}{r-3} \frac{4}{r-2} D_{r-1} \\
&+ 4 \binom{n-4}{r-3} \Big(\frac{2}{r-1} D_r - D_{r-2} - D_{r-3} \Big).
\end{aligned}
$$

Moreover, we also have:

$$
\begin{aligned}
I(n,2,r) &= \sum_{t=0}^{r-3} \left(\binom{n-4}{t} + \binom{n-4}{t-1} \right) \Big(D_t + 3D_{t+1} + 3D_{t+2} + D_{t+3} \Big) \\
&\quad + \binom{n-3}{r-2} \Big(D_{r-2} + 2D_{r-1} + D_r \Big) \\
&= \sum_{t=0}^{r-4} \binom{n-4}{t} \Big(D_t + 4D_{t+1} + 6D_{t+2} + 4D_{t+3} + D_{t+4} \Big) + \Delta_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_1 = & \binom{n-4}{r-3} D_{r-3} + \left(3\binom{n-4}{r-3} + \binom{n-3}{r-2} \right) D_{r-2} + \left(3\binom{n-4}{r-3} + 2\binom{n-3}{r-2} \right) D_{r-1} \\
&+ \left(\binom{n-4}{r-3} + \binom{n-3}{r-2} \right) D_r .
\end{aligned}
$$

Since $r \geqslant 5$ then we have $\frac{4}{r-2} < 2$ and $\frac{8}{r-1} \leqslant 2$. Therefore, we have:

$$
\Delta \leqslant 2\binom{n-4}{r-2} D_{r-2} + 2\binom{n-4}{r-3} D_{r-1} + 2\binom{n-4}{r-3} D_r < \Delta_1,
$$

since $D_{r-2} < D_{r-1}$ and $\binom{n-4}{r-3} \leqslant \binom{n-3}{r-2}$. So, it follows that for any $r \geqslant 5$ we have $I(n,2,r) > I(n,4,r)$. Moreover, by the definition $N(n,r)$ we have $N(n,r) \geqslant I(n,2,r) = \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2} \big)$ for any $r \geqslant 5$. □

Now let us consider a probability of reconstructing permutations from at most four distinct permutations in $B_2(\pi)$. By Theorem 4 we have $N(n,2) = 3$ for any $n \geqslant 3$. We denote by t_i the number of sets of i distinct permutations belonging to $B_2(\pi)$ from which $\pi \in \mathrm{Sym}_n$ is reconstructible. Let us also denote

$$
M = B_2(n) = 1 + \binom{n}{2} = \frac{n^2 - n + 2}{2}
$$

and consider the probability $p_i = \frac{t_i}{\binom{M}{i}}$ of the event that a permutation $\pi \in \mathrm{Sym}_n$ is reconstructbile from i distinct permutations in $B_2(\pi)$ under the condition that these permutations in $B_2(\pi)$ are uniformly distributed. Without loss of generality we let $\pi = I_n$. If we choose an element $\hat{\pi} \in B_2(I_n)$ then $\hat{\pi} = I_n$ or $\hat{\pi} = (i\ j)$ with $i \neq j$ for some $i, j \in [n]$. Assume we take an element $\hat{\pi}$ from $B_2(\pi)$, regardless of whether $\hat{\pi} = I_n$ or $\hat{\pi} = (i\ j)$, we have that $\{I_n, (i\ j)\} \subseteq B_2(I_n) \cap B_2((i\ j))$. Therefore, we cannot use I_n or $(i\ j)$ to distinguish I_n and $(i\ j)$. So, $p_1 = 0$. Similarly, assume we choose two elements $\pi^{(1)}, \pi^{(2)}$ we prove that $p_2 = 0$. When $\pi^{(1)} = I_n$ or $\pi^{(2)} = I_n$, then $\{\pi^{(1)}, \pi^{(2)}\} = \{I_n, (i\ j)\}$ with $i \neq j$ for some $i, j \in [n]$. Since $\{I_n, (i\ j)\} \subseteq B_2(I_n) \cap B_2((i\ j))$ then we cannot use $\{I_n, (i\ j)\}$ to distinguish I_n and $(i\ j)$. When $\pi^{(1)} = (i\ j)$ and $\pi^{(2)} = (j\ k)$ with $i \neq j \neq k$ for some $i, j, k \in [n]$, then $\{(i\ j), (j\ k)\} \subseteq B_2(I_n) \cap B_2((i\ j\ k))$. Thus, we cannot use $(i\ j)$ and $(j\ k)$ to distinguish I_n and $(i\ j\ k)$. When $\pi^{(1)} = (i\ j)$ and $\pi^{(2)} = (k\ l)$ with $i \neq j \neq k \neq l$ for some $i, j, k, l \in [n]$, then $\{(i\ j), (k\ l)\} \subseteq B_2(I_n) \cap B_2((i\ j)(k\ l))$. Thus, we cannot use $(i\ j)$ and $(k\ l)$ to distinguish I_n and $(i\ j)(k\ l)$. So, $p_2 = 0$. Therefore, we can never reconstruct any permutation $\pi \in \mathrm{Sym}_n$ from a single permutation or from two distinct permutations in $B_2(\pi)$. It is obvious that $p_4 = 1$ which means that we can always reconstruct any permutation $\pi \in \mathrm{Sym}_n$ from any four distinct permutations in $B_2(\pi)$.

Lemma 16 $\lim\limits_{n\to\infty} p_3 = 1$.

Proof By the definition, $p_3 = \frac{t_3}{\binom{M}{3}}$, where t_3 is the number of sets of three distinct permutations in $B_2(\pi)$ from which $\pi \in \mathrm{Sym}_n$ is reconstructible. Without loss of generality we use I_n instead of π. By the proof of Theorem 2, we have $B_2(I_n) \cap B_2(\sigma) = \{(i\ j), (i\ k), (j\ k)\}$ for any permutation $\sigma = (i\ j\ k) \in S_3(I_n)$ and for any three distinct integers $i, j, k \in [n]$. Moreover, $|B_2(I_n) \cap B_2(\sigma)| \leqslant 2$ for any permutation $\sigma \notin S_3(I_n)$ since $I(n, 2, 2) = 2$, $I(n, 4, 2) = 2$, and $I(n, d, 2) = 0$ for any $d \geqslant 5$. This means that I_n is not reconstructible only in the case if we take three permutations from the set $\{(i\ j), (i\ k), (j\ k)\}$ for any distinct integers $i, j, k \in [n]$. Therefore, we have:

$$t_3 = \binom{M}{3} - \binom{n}{3} = \frac{n^6 - 3n^5 + 3n^4 - 9n^3 + 20n^2 - 12n}{48},$$

where $\binom{M}{3} = \frac{n^6 - 3n^5 + 3n^4 - n^3 - 4n^2 + 4n}{48}$. This immediately gives us

$$\lim_{n\to\infty} p_3 = \lim_{n\to\infty} \frac{t_3}{\binom{M}{3}} = 1.$$

Hence, this completes the proof. □

Now we determine $N(n, r)$ for arbitrary $r \geqslant 2$ and sufficiently large n and give the asymptotic property of $N(n, r)$ for any $r \geqslant 2$. We start with studying $I(n, d, r)$ for sufficiently large n. By the definition of D_r and Lemma 6 we have $D_r = r \cdot D_{r-1} + (-1)^r$ for any $r \geqslant 1$. Combing with $D_1 = 0$, $D_2 = 1$, and $D_3 = 2$, it is easily verified that $D_r < r!$ for any $r \geqslant 1$.

Lemma 17 *For given integers r, d such that $5 \leqslant d \leqslant 2r$ and for sufficiently large n, the following holds:*

$$I(n, d, r) = \Theta(n^{r - \lceil \frac{d}{2} \rceil}). \tag{47}$$

Proof By Lemma 3 there exists a permutation $\sigma \in \mathrm{Sym}_n$ such that $I(n, d, r) = |B_r(I_n) \cap B_r(\sigma)|$ where $d(I_n, \sigma) = d$ and $\sigma(i) = i$ for $d + 1 \leqslant i \leqslant n$.

Let $\alpha \in B_r(\sigma)$ and let us check whether $\alpha \in B_r(I_n)$. We put $\alpha = [\alpha_1 \dots \alpha_d\, \alpha_{d+1} \dots \alpha_n] = [a_1\, a_2]$, where $a_1 = [\alpha_1 \dots \alpha_d]$, $a_2 = [\alpha_{n-d} \dots \alpha_n]$. In a similar way, we consider $I_n = [e_1\, e_2]$ and $\sigma = [s_1\, s_2]$, where $e_1 = [1 \dots d]$, $s_1 = [\sigma_1 \dots \sigma_d]$, and $e_2 = [d+1 \dots n] = s_2$.

Let us show that $I(n,d,r) \leqslant O(n^{r-\lceil \frac{d}{2} \rceil})$. We assume that $d(a_1, s_1) = i$ and $d(a_2, s_2) = j$ for some $0 \leqslant i + j \leqslant r$. By this assumption, we have $d(\alpha, \sigma) = i + j$ and $s_2 = e_2$ which gives us $d(a_2, e_2) = j$. Let $k = \lceil \frac{d}{2} \rceil$. Due to the lemma conditions we have $r \geqslant k$. If $0 \leqslant j \leqslant r - k$ then we have:

$$|\{\alpha \in B_r(\sigma) | d(a_2, s_2) = j\}| = \sum_{i=0}^{r-j} \binom{d}{i} \binom{n-d}{j} D_{i+j}. \tag{48}$$

If $r-k+1 \leqslant j \leqslant r$ then $0 \leqslant d(a_1, s_1) \leqslant k-1$. Since $d(s_1, e_1) = 2k-1$ or $d(s_1, e_1) = 2k$ then we have $d(a_1, e_1) \geqslant k$. Thus, it follows:

$$d(\alpha, I_n) = d(a_1, e_1) + d(a_2, e_2) \geqslant k + j \geqslant r + 1,$$

for some fixed j such that $r - k + 1 \leqslant j \leqslant r$. Hence, for any $\alpha \in B_r(\sigma)$ such that $d(a_2, s_2) = j$ and $j \geqslant r - k + 1$ we have $\alpha \notin B_r(I_n) \cap B_r(\sigma)$. Then the following holds:

$$\begin{aligned} I(n,d,r) = |B_r(I_n) \cap B_r(\sigma)| &\overset{a}{\leqslant} \sum_{j=0}^{r-k} \sum_{i=0}^{r-j} \binom{d}{i} \binom{n-d}{j} D_{i+j} \\ &\overset{b}{\leqslant} \sum_{j=0}^{r-k} \sum_{i=0}^{r-j} \binom{d}{i} \binom{n-d}{j} (i+j)! \\ &\overset{c}{\leqslant} O(n^{r-k}), \end{aligned} \tag{49}$$

where $\overset{a}{\leqslant}$ follows from (48), $\overset{b}{\leqslant}$ follows from the fact that $D_r \leqslant r!$ for any $r \geqslant 1$, and $\overset{c}{\leqslant}$ follows from $\binom{d}{i}\binom{n-d}{j}(i+j)! \leqslant O(n^j)$ and $j \leqslant r - k$.

Now we prove that $I(n,d,r) \geqslant \Omega(n^{r-\lceil \frac{d}{2} \rceil})$ under the same conditions on d. If $d = 2k-1$ then $k \geqslant 3$. Let $\sigma = (1\,2 \dots k-1)(k\,k+1 \dots 2k-1)$ and $V = \{\alpha \in \mathrm{Sym}_n | \alpha_1 = (1\,2 \dots k-1),\ d(a_2, s_2) = r - k\}$.

For any $\alpha \in V$, we have $d(\alpha, \sigma) = d(a_1, s_1) + d(a_2, s_2) = k + r - k = r$ and $d(\alpha, I_n) = d(a_1, e_1) + d(a_2, e_2) = k - 1 + r - k = r - 1$. This means that $V \subset B_r(I_n) \cap B_r(\sigma)$ and we have:

$$|V| = \binom{n-2k+1}{r-k}.$$

Therefore, if $d = 2k - 1$ and $5 \leqslant d \leqslant 2r$ we have:

$$I(n,d,r) \geqslant |B_r(I_n) \cap B_r(\sigma)| \geqslant |V| = \Omega(n^{r-k}).$$

Similarly, if $d = 2k$ then let us put $\sigma = (1\,2 \dots k)(k+1\,k+2 \dots 2k)$ and let $V_1 = \{\alpha \in \mathrm{Sym}_n | a_1 = (1\,2 \dots k),\ d(a_2, s_2) = r - k\}$. Then for any $\alpha \in V_1$ we have $d(\alpha, \sigma) = d(a_1, s_1) + d(a_2, s_2) = k + r - k = r$ and $d(\alpha, I_n) = d(a_1, e_1) + d(a_2, e_2) = k + r - k = r$. Thus, $V_1 \subset B_r(I_n) \cap B_r(\sigma)$. Moreover, we have

$$|V_1| = \binom{n-2k}{r-k}.$$

Therefore, if $d = 2k$ and $5 \leqslant d \leqslant 2r$ we have:

$$I(n, d, r) \geqslant |B_r(I_n) \cap B_r(\sigma)| \geqslant |V_1| = \Omega(n^{r-k}).$$

Thus, taking into account the cases above we have $I(n, d, r) \geqslant \Omega(n^{r-\lceil \frac{d}{2} \rceil})$ for any $5 \leqslant d \leqslant 2r$ which completes the proof. □

Theorem 9 *Let $r \geqslant 3$. Then for sufficiently large n we have:*

$$N(n, r) = \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2}\big).$$

Moreover, $N(n, r) = \Theta(n^{r-2})$.

Proof If $r = 3, 4$ then from (11),(12) and by Theorems 5,6 we have:

$$N(n, r) = I(n, 2, r) = \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2}\big).$$

If $r \geqslant 5$ then by Lemma 17 we have $I(n, d, r) = \Theta(n^{r-\lceil \frac{d}{2} \rceil})$ for any $5 \leqslant d \leqslant 2r$ which means $r - \lceil \frac{d}{2} \rceil \leqslant r - 3$ and $I(n, d, r) = o(n^{r-2})$ in this case.

By Theorem 8, if $2 \leqslant d \leqslant 4$ then $I(n, 2, r) \geqslant \max\{I(n, 3, r), I(n, 4, r)\}$. Moreover, we have $I(n, 2, r) = \Theta(n^{r-2})$. By the definition of $N(n, r)$, we immediately have:

$$N(n, r) = \max_{l \geqslant 1} I(n, l, r) = I(n, 2, r)$$

for any $r \geqslant 5$ and sufficiently large n.

Therefore, for any $r \geqslant 3$ and sufficiently large n, we have:

$$N(n, r) = I(n, 2, r) = \sum_{t=0}^{r-2} \binom{n-2}{t} \big(D_t + 2D_{t+1} + D_{t+2}\big).$$

Furthermore, we have $N(n, r) = \Theta(n^{r-2})$ for any $r \geqslant 3$. □

6 Conclusion

In this paper, the sequence reconstruction problem is studied over permutations with the Hamming metric for $r = 2, 3, 4$ or sufficiently large n. Since $N(n, r) = I(n, 2, r)$ for $r = 3, 4$, and for any $r \geqslant 5$ and sufficiently large n, there is the following conjecture.

Conjecture 1 $N(n, r) = I(n, 2, r)$ for any $r \geqslant 5$ and $n \geqslant r$.

Furthermore, if the transmitted sequence is a codeword in Sym_n with Hamming distance d and with at most r errors over each channel then the following sequence reconstruction problem is stated.

Problem 1 Determine

$$N(n, d, r) = \max_{\pi, \tau \in \mathrm{Sym}_n, d(\pi, \tau) \geqslant d} |B_r(\pi) \cap B_r(\tau)| = \max_{l \geqslant d} I(n, l, r).$$

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Declarations

Conflict of interest The authors declare that there is neither conflict of interest nor conflict of interest.

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